Exploring Properties of Quadrilaterals in the Poincaré Disk Model of Hyperbolic Geometry Using the Dynamic Geometry Software

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The purpose of this study was to have preservice mathematics teachers explore important properties of hyperbolic quadrilaterals in the Poincaré disk model using the Dynamic Geometry Software (DGS). The participants comprised 40 preservice mathematics teachers at Rajabhat Rajanagarindra University, Thailand, who had enrolled in the Foundations of Geometry course at the second semester of the academic year 2015. The instruments were activity packages exploring properties of hyperbolic quadrilaterals using DGS. The results indicated that preservice mathematics teachers could make conjectures and verify properties of hyperbolic quadrilaterals correctly and rapidly. The use of DGS can help students visualize this abstract geometry.

INTRODUCTION

Among the branches of mathematics, geometry has been most subject to changing tastes from age to age (Merzbach & Boyer, 2011, p.483). It is a classic mathematics subject because of two reasons, i.e. it had the origin and developments in classical times and most students in high schools are introduced to study geometry (Lezark & Capaldi, 2016, p.1). Etymologically, the word "geometry" is derived from two words "earth" (geo) and "measure" (metry). The idea of earth measure was significant in the ancient, pre–Greek development of geometry (Smart, 1998, p.1). In Thailand, only Euclidean geometry is studied in primary and secondary levels. Non-Euclidean geometries are differently covered on at the universities level for students majoring in mathematics or mathematic education. At the college level, geometry is still a difficult course for most students because it indispensably requires them to reason strictly from axioms and postulates rather than informal experiences and intuitive understandings. In order to enable students to appreciate the importance of the rigorous axiomatic approach, most college geometry courses introduce students to a less intuitive world of non–Euclidean geometry. Generally, students enter a college geometry course with twelve or more years of experience working within the Euclidean system of axioms. By this, students' understanding of figures and relationships within this system is challenged when the axioms are modified (Smith, Hollebrands, Iwancio, & Kogan, 2007, p. 613). While geometry, in general, is a very visual subject, there are several limitations to students' uses of paper–and–pencil diagrams, especially when it comes to non–Euclidean geometries. A student may create inaccurate misleading diagrams and arrive to incorrect conjectures. Also, a student may create a correct diagram that is too specific; this may inhabit students' ability to derive general conclusions and proofs that go beyond the drawing they have create (Schoenfeld, 1986, pp. 225–264).

Many researchers, mathematics educators, and professional organizations have suggested the use of dynamic software programs to teach geometry e.g. Geometer's Sketchpad, GeoGebra, Cabri. These software programs enable students to construct creatively an accurate diagram and to interact with the diagrams in order to abstract general properties and relationship because the ways in which the programs respond to the students' actions is determined by geometrical theorems. Relevant research studies have been shown to improve their understanding of geometrical concepts and support their development of formal proofs (Groman, 1996; Singmuang & Phahanich, 2004; Laborde, Kynigos, Hollenbrands & Straesser, 2006; Smith, Hollebrands, Iwancio, & Kogan, 2007; Myers, 2009; Kurtuluş & Ada, 2011; Singmuang, 2013; Bhagat & Chang, 2015; Lorsong & Singmuang, 2015; Yilmaz, 2015; Singmuang, 2016; Bist, 2017; Sebial, 2017). Apparently, various technological tools have been developed to facilitate students in reasoning within different non–Euclidean systems such as Geometers' sketchpad, NonEuclid, GeoGebra, Cabri, but little research has examined how students' uses of the Geometry Explorer affects their understandings of properties of quadrilaterals in hyperbolic geometry.

NON–EUCLIDEAN GEOMETRY

Classical Euclidean geometry is an axiomatic system. In an axiomatic system, one proves statements based on a set of agreed–upon statements called axioms, or postulates, which need no proof. Logically, one needs axioms in order to avoid an infinite regression of statements which depends on other statements. Axioms are supposed to be fairly self–evident and obvious to those working in the system. In Euclid's axiomatic system there are five postulates:

1. Between any two distinct points, a segment can be constructed.
2. Segments can be extended indefinitely.
3. Given a point and a distance, a circle can be constructed with the point as center and the distance as radius.
4. All right angles are congruent.
5. Given two lines in the plane, if a third line m crosses the given lines such that the two interior angles on one side of m are less than two right angles, then the two lines if continued indefinitely will meet on that side of m where the angles are less than two right angles.

Postulates 1–4 seem very intuitive and self–evident. The fifth postulate, the so–called parallel postulate, seems overly complex for an axiom. It is not self–evident or obvious and reads more like a theorem. Euclid seems to have recognized this himself, since he postponed using it as long as possible. The struggle to prove Euclid's fifth postulate continued well into the 18^{th} century, but eventually mathematicians realized that the fifth postulate is independent of the first four. In other words, there exist geometries in which the negation of the fifth postulate is an axiom. These geometries came to be known as non–Euclidean geometries–hyperbolic and elliptic. Felix Klein is credited with giving the names *hyperbolic, elliptic*, and *parabolic*, in 1871, to two non–Euclidean geometries and to Euclidean geometry, respectively (Smart, 1998, p.368). Non–Euclidean geometry provided Einstein with a suitable model for his work on relativity. Although it also occurs in differential geometry and elsewhere, it is worthwhile for other reasons as well (Smart, 1998, p.367).

HYPERBOLIC GEOMETRY

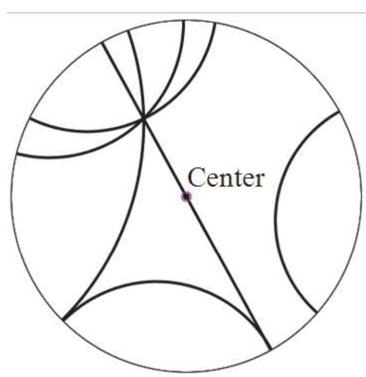
The discovery of non–Euclidean geometry is one of the most important events in the history of mathematics. Credit for the discovery of hyperbolic geometry is generally given instead to the Russian mathematician, Nicolai Ivanovich Lobachevsky (1793–1856), and the Hungarian mathematician, János Bolyai (1802–1860), who published their independent work in 1829 and 1832, respectively. The eminent mathematician, Karl Friedrich Gauss (1777–1855) also worked extensively in hyperbolic geometry but left his results unpublished (Cederberg, 2001, p. 48). Nowadays, the many applications of non–Euclidean geometries range from statistics to relativity. Hyperbolic geometry is usually introduced to students as a first example of a non–Euclidean geometry because it is the "closest" one to Euclidean geometry. By this, it involved changing only one axiom. But this one change in an axiom has surprising implications (Dwyer & Pfiefer, 1999, p. 632). We obtain hyperbolic geometry by keeping all the axioms of Euclidean geometry except the parallel postulate and replacing it with the hyperbolic parallel postulate. The hyperbolic parallel postulate asserts that for every line n and for every point P that does not lie on n, there exist multiple lines through P that are parallel to n.

Poincaré Disk Model for Hyperbolic Geometry

A model for a geometry is an interpretation of the technical terms of the geometry (such as point, line, distance, angle measure etc.) that is consistent with the axioms of the geometry (Venema, 2003, p. 111). There are many ways in which models of hyperbolic geometry is constructed. Some of these models are the Klein model, Maximum Plan model, Poincaré Upper Half–Plane model, and Poincaré disk modle. Significantly, the Poincaré disk model is probably the most popular model of hyperbolic geometry. One reason for its popularity is the great beauty of the diagrams that can be constructed in it. This model is named for Henry Poincaré (1854–1912) because he was the first French mathematician who introduced it and described this model in 1882. This study was conducted by using hyperbolic geometry and Poincaré disk model.

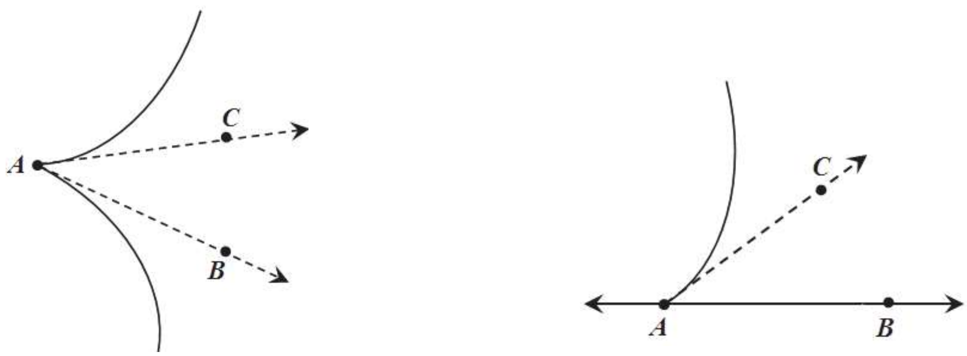
In the Poincaré model for 2–dimensional hyperbolic geometry, a point is defined to be any point interior to the unit disk. That is, any point P = (x, y), with $x^2 + y^2 < 1$. The collection of all such points will be called the Poincaré disk. The Poincaré disk model is technically a Euclidean model of hyperbolic geometry because we begin with a circle, in the Euclidean plane, then define all the points in the interior of the circle to be the points of our model. The points on and outside the circle are still Euclidean and still exits, but the hyperbolic geometry world lies only inside the circle. Lines in this geometry are quite different than Euclidean lines. The lines of the model are defined to be the arc of (Euclidean) circles that meet the boundary of the given circle at right angles, and straight-line segments through the center of the given circle. Therefore, there are two kinds of "lines" in the Poincaré disk (see Figure 1).

FIGURE 1
THE LINES OF HYPERBOLIC PLANE



The angle between two Poincaré lines is measured by measuring the Euclidean angle between their tangent lines as shown in Figure 2(a) and (b). This Poincaré disk model is popular because of this feature. The Poincaré model preserves the Euclidean notion of angle, but at the expense of defining lines in a fairly strange manner.

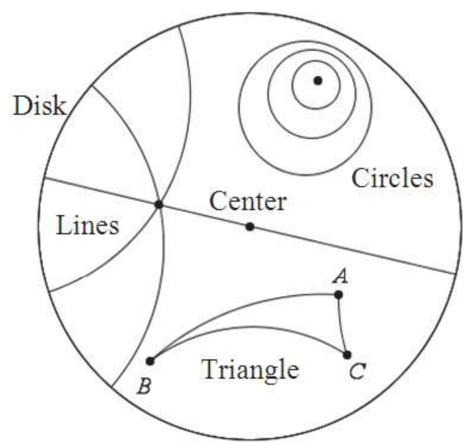
FIGURE 2
ANGLE BETWEEN TWO ARCS OR AN ARC AND A LINE



a b

As we go inside our hyperbolic world, most other geometric objects will be defined in the usual way. For example, triangles will still be defined as the three hyperbolic line segments joining three (hyperbolic) noncollinear points, *A, B*, and *C*, and will be denoted ΔABC (Figure 3).

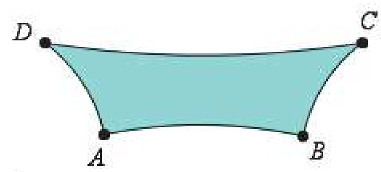
FIGURE 3
VARIOUS GEOEMTRIC OBJECTS IN THE POINCARÉ DISK



Saccheri Quadrilateral

Giovanni Girolamo Saccheri (1667–1733), an Italian Jesuit priest, played a pivotal role in clarifying the questions related to Euclid's fifth postulate. His idea was to take as an axiom the negation of Euclid's fifth postulate and to develop from there a geometric theory, with the hope of eventually reaching a contradiction, thus proving the validity of Euclid's fifth postulate (Borceux, 2014, p.251). Interestingly, Saccheri published a book entitled "*Euclid Freed of Every Flaw*", not long before he died in 1733. In his book, he summarized on negating the parallel postulate by the "hypothesis of the acute angle" (Hvidsten, 2005, p. 286)**.** Saccheri's idea was to study quadrilaterals whose base angles are right angles and whose base–adjacent sides are congruent. Definitely, in Euclidean geometry, such quadrilaterals must be rectangles; that is, the top (or summit) angles must be right angles. Saccheri negated the parallel postulate by assuming the summit angles were less than 90 degrees. This was the "hypothesis of the acute angle." Saccheri's attempt to prove the parallel postulate ultimately failed because he could not find a contradiction to the acute angle hypothesis (Hvidsten, 2017, p. 296)**.** In his attempt to prove the fifth postulate, Saccheri made use of a set of points now called a Saccheri quadrilateral. A Saccheri quadrilateral has two right angles and two congruent sides, as shown in Figure 4, segment *AB* is called the base and segment *CD* is called the summit of the quadrilateral. The two congruent segments are the sides.

FIGURE 4
A SACCHERI QUADRILATERAL *ABCD* WITH ∠*A* AND ∠*B* ARE RIGHT ANGLES



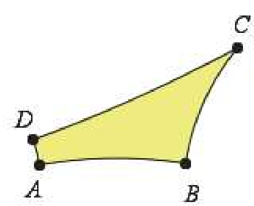
Some of the properties of the Saccheri quadrilateral are as follows:

1. The summit angles of a Saccheri quadrilateral are always congruent.
2. The summit angles of a Saccheri quadrilateral are always acute.
3. The segment joining the midpoints of the base and summit of a Saccheri quadrilateral makes right angles with the base and summit (is perpendicular to both).

Lambert Quadrilateral

Johann Heinrich Lambert (1728–1777), like Saccheri, attempted to prove the fifth postulate by an indirect argument. He began with a quadrilateral with three right angles, now called a Lambert quadrilateral, shown in Figure 5.

FIGURE 5
A LAMBERT QUADRILATERAL *ABCD* WITH RIGHT ANGLES AT ∠*A*, ∠*B*, AND ∠*D*



Some properties of the Lambert quadrilateral are as follows:

1. The fourth angle (the one not specified to be a right angel) must be acute.
2. The sides adjoining the fourth angle (the one not specified to be a right angel) are greater than the opposite sides.

A Lambert quadrilateral can be constructed from a Saccheri quadrilateral by joining the midpoints of the base and summit of the Saccheri quadrilateral. This line segment is perpendicular to both the base and summit and so either half of the Saccheri quadrilateral is a Lambert quadrilateral.

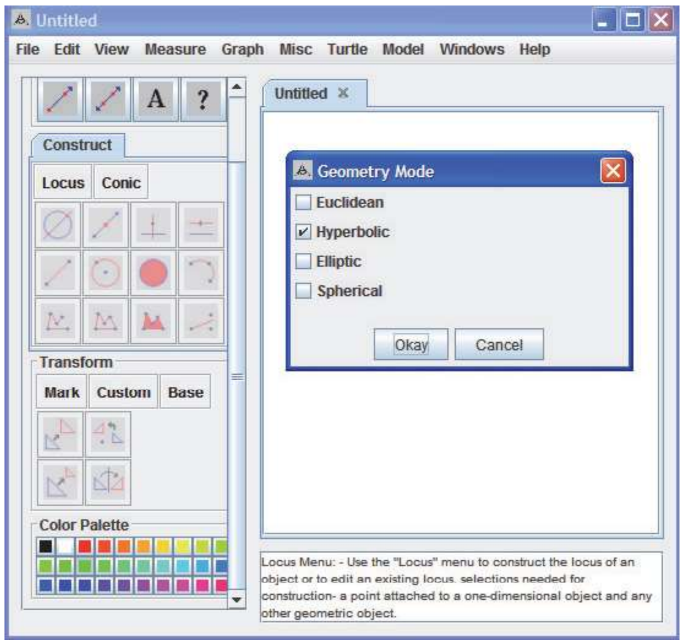
GEOMETRY EXPLORER

Main Geometry Explorer Window

Geometry Explorer is designed as a geometry laboratory where one can create geometric objects (like points, circles, polygons, areas, and the like, carry out transformations on these objects (dilations, reflections, rotations, and translations), and measure aspects of these objects (like length, area, radius, and so on). In this case, it is much like doing geometry on paper (or sand) with a ruler and compass. However, on paper such constructions are static–points placed on the paper can never be moved again. In Geometry Explorer, all constructions are dynamic. One can draw a segment and then grab one of the endpoints and move it around the canvas with the segment moving accordingly. Thus, one can create a construction and test out hypotheses about construction with numerous variations of the original construction. Geometry Explorer is just what the name implies–an environment to explore geometry (Hvidsten, 2005, p. 411).

Upon starting Geometry Explorer you will see the main Geometry Explorer window on the screen (Figure 6).

FIGURE 6
THE MAIN GEOMETRY EXPLORER WINDOW



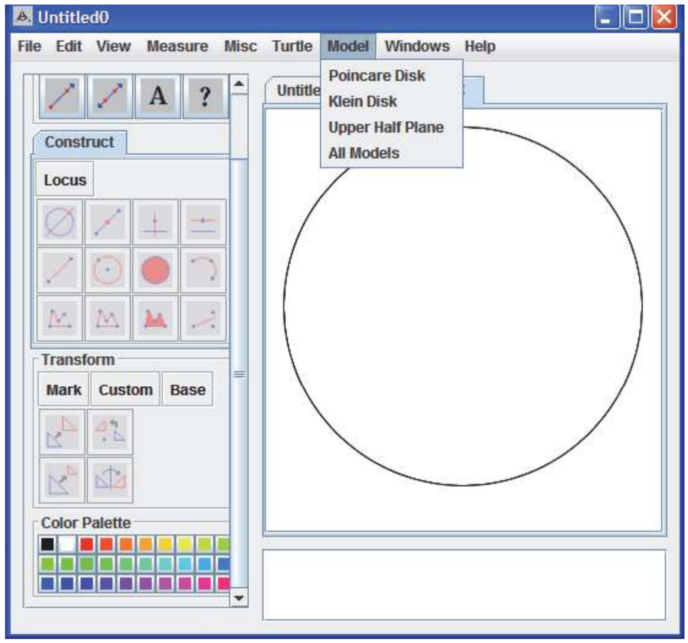
There are four important areas within this main window.

1. The *Canvas* where geometry is created and transformed. This is the large white area on the right side of the main window.
2. The *Tool Panel* where geometric tools are located. The Tool Panel is directly to the left of the Canvas. It consists of a set of iconic buttons which represent tools used to create and modify geometric figures. The icons (pictures) on the buttons depict the function that the particular button serves. This function is sometime quite clear. Other times it is harder to figure out, but the pictures serve as reminders as to what the buttons can do. The Tool Panel is split into four sub-panels: *Create, Construct, Transform*, and *Color Palette*.
3. The *Menu Bar*. There are 10 menus shown in the menu bar: *File, Edit, View, Measure, Graph, Misc, Turtle, Model, Windows,* and *Help*. Each menu will control specific actions.
4. The *Message Box*. This is where detailed information will be shown concerning various tools that one may wish to use. In the Message Box we see information concerning how this tool should be used, as well as other information provided by the tool. In the case of the Info tool, we see information regarding memory use for the program. The Message Box is located below the Canvas.

Working in the Hyperbolic Canvas

When one opens a new Geometry Explorer window (using the *New* menu option under the File menu) a dialog box pops up asking which of the four geometries–Euclidean, Hyperbolic, Elliptic, or Spherical–will be used in the new window. To start working in hyperbolic geometry clicks on *"Hyperbolic"* in the dialog box and click Okay. A Geometry Explorer window will pop up with a view of the Poincaré disk model of hyperbolic geometry in the Canvas (Figure 7). The Poincaré disk model is the default mode used by Geometry Explorer. In order to switch to the other models, just choose one of three options listed under the Model menu in the main window.

FIGURE 7
THE HYPERBOLIC WORKSPACE MAIN WINDOW



Geometry Explorer employs three different models of hyperbolic geometry, the Poincaré disk model, the Klein disk model, and the Upper Haif–Plane model, as environments for the exploration of the "strange new universe" of non-Euclidean geometry.

METHODOLOGY

This study was aimed to have Thai preservice mathematics teachers explore some of the properties of the Saccheri quadrilateral and the Lambert quadrilateral in the Poincaré Disk Model of hyperbolic geometry using the Geometry Explorer program.

The participants in this study were 40 Thai preservice mathematics teachers at Rajabhat Rajanagarindra University, Thailand, who had enrolled in the Foundations of Geometry course at the second semester of the academic year 2015. They were selected by using cluster random sampling method. After being introduced to the technical properties of Geometry Explorer, the sampled students were asked to complete the activities by using Geometry Explorer tools so that they explored the hyperbolic geometry modeled by Poincaré disk model. The research instruments used in this study were activity packages exploring properties of Saccheri quadrilateral and Lambert quadrilateral in hyperbolic geometry using the Geometry Explorer program.

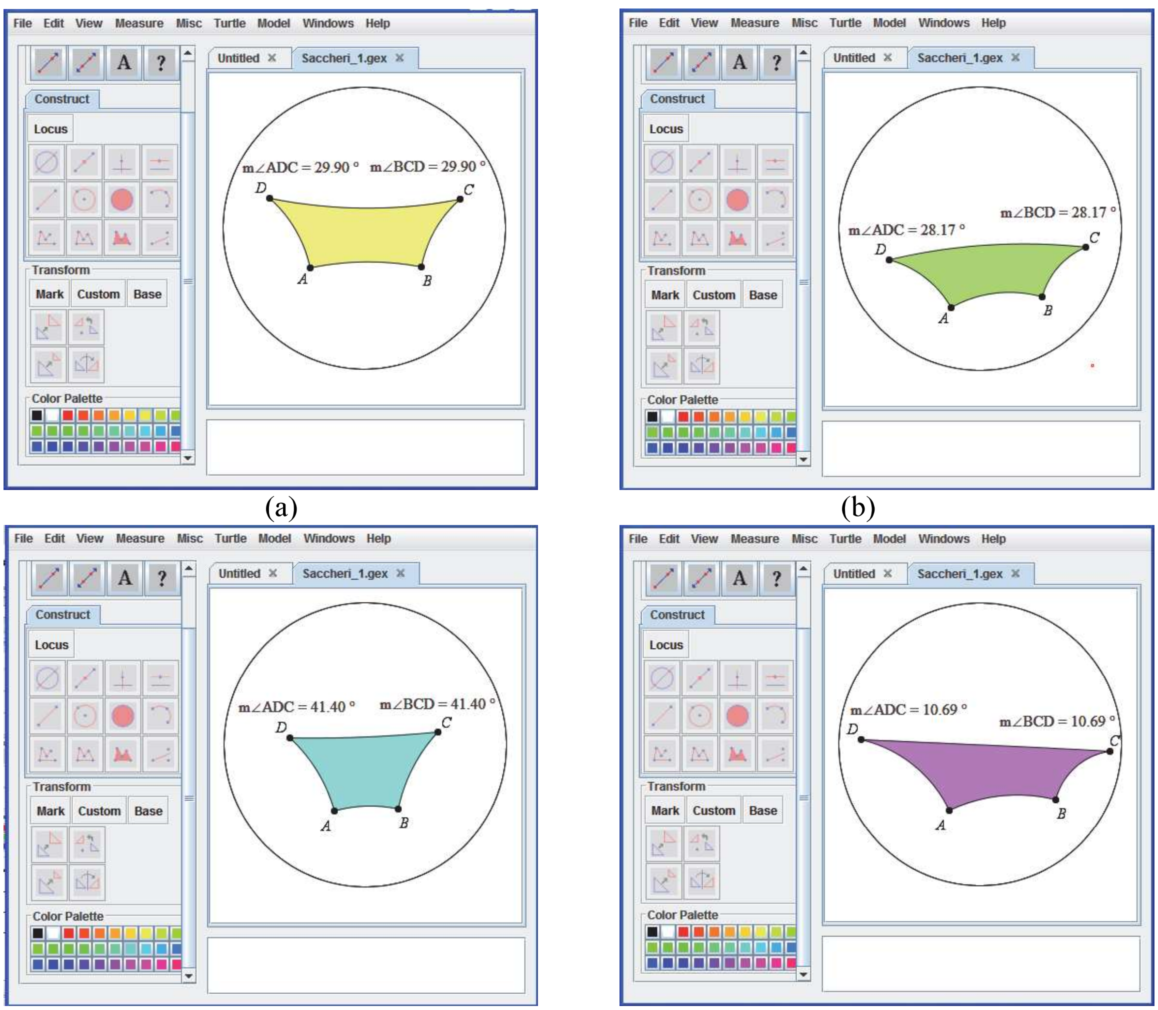
RESULTS

After introducing the basic concepts of hyperbolic geometry, such as point, line, angle, circle, perpendicular line, definition of a Saccheri quadrilateral, definition of a Lambert quadrilateral to the students by using the Geometry Explorer, Thai preservice mathematics teachers began their exploration activities in the computer-based environment.

Exploration 1: The Summit Angles of a Saccheri Quadrilateral are Equal.

The students were assigned to construct a Saccheri quadrilateral by using the Geometry Explorer software. After that, they were asked to measure the two summit angles in the quad and compare their measures. Then, they were asked to make observations for different Saccheri quadrilaterals by dragging their first quadrilateral. Students made some observation by following the directions. In a short time period, most of the students realized that the two summit angles are equal. Some students tested their conjectures for different Saccheri quadrilaterals, as seen in Figure 8(a), (b), (c), and (d). Other students also confirmed this result by making their observations with their own Saccheri quadrilaterals.

FIGURE 8
SACCHERI QUADRILATERALS WITH ∠*C* AND ∠*D* ARE EQUAL

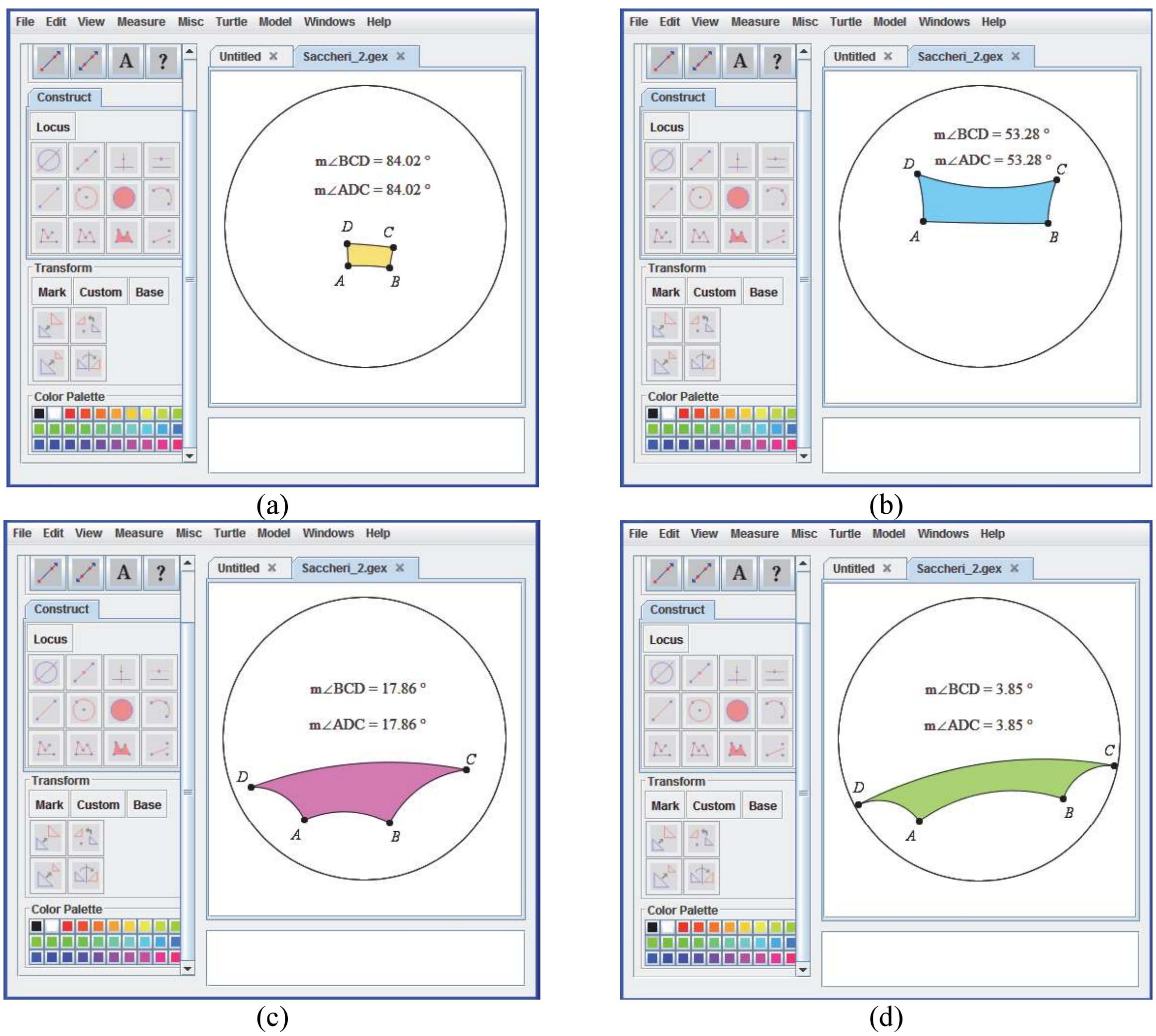


(a) (b) (c) (d)

Exploration 2: The Summit Angles of a Saccheri Quadrilateral are Acute.

The researcher asked the students to construct another Saccheri quadrilateral by using the Geometry Explorer software, measure the two summit angles, $\angle C$ and $\angle D$, in the quad, and observe the type of those summit angles whether they are acute, obtuse, or right angles. The students were asked to make observations for different Saccheri quadrilaterals by dragging their first quadrilateral. Students made some observation by following the directions. After their observations on the numerical values, most of the students found that the measure of $\angle C$ was less than 90 degrees and the measure of $\angle D$ was also less than 90 degrees as shown in Figure 7(a), (b), (c), and (d). Therefore, they concluded that the two summit angles are acute angles.

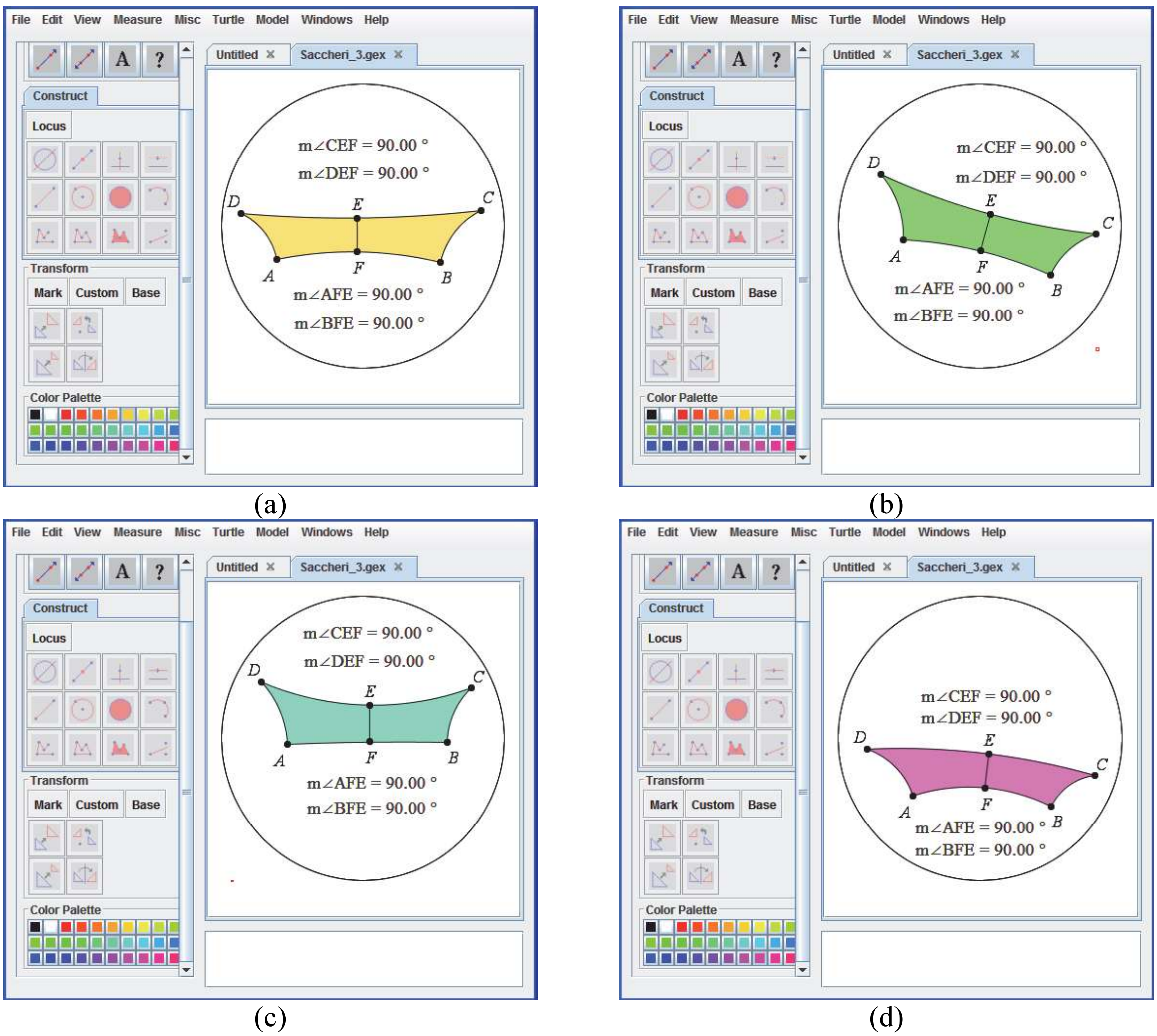
FIGURE 7
SACCHERI QUADRILATERALS WITH $\angle C$ AND $\angle D$ ARE ACUTE ANGLES



Exploration 3: The Segment Joining the Midpoints of the Base and Summit of a Saccheri Quadrilateral Makes Right Angles with the Base and Summit (is Perpendicular to Both).

The students were asked to construct another Saccheri quadrilateral by using the Geometry Explorer software, create the midpoint of both the base and the summit of a Saccheri quadrilateral, and draw the line joining the midpoints of the base and the summit. The students were told that this line is called the altitude of the Saccheri quadrilateral. Then, they were asked to measure the angle between the altitude and the base and the angle between the altitude and the summit. The students were asked to observe those types of angles. After the observations, most of the students concluded that segment joining the midpoints of the base and summit of a Saccheri quadrilateral made right angles with the base and summit. It was perpendicular to both as shown in Figure 8(a), (b), (c), and (d).

FIGURE 8
THE SEGMENT JOINING THE MIDPOINTS OF THE BASE AND SUMMIT OF A QUADRILATERAL MAKES RIGHT ANGLES WITH THE BASE AND SUMMIT



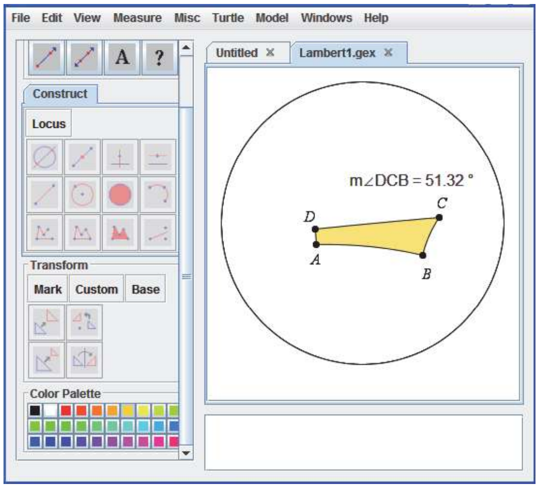
Exploration 4: In a Lambert Quadrilateral the Fourth Angle (the One not Specified to be a Right Angel) Must be Acute.

The researcher asked the students to draw a Lambert quadrilateral *ABCD* with right angles at $\angle A$, $\angle B$, and $\angle D$. Then, the researcher asked the students to measure the angle, not specified to be a right angel, $\angle C$.

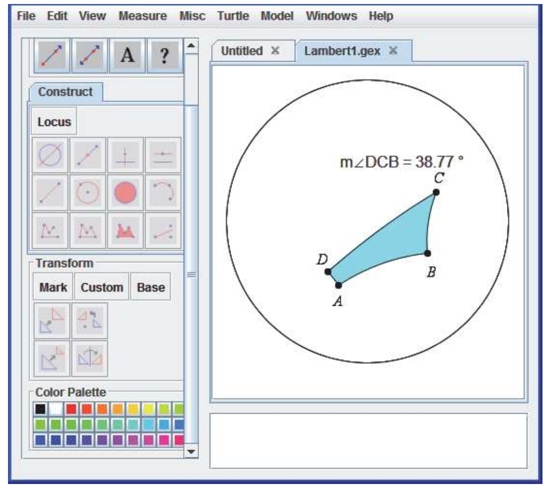
After their observations on the numerical values (Figure 9(a), (b), (c), and (d)), most of the students attained the following conjecture:

'The fourth angle in a Lambert quadrilateral is acute.'

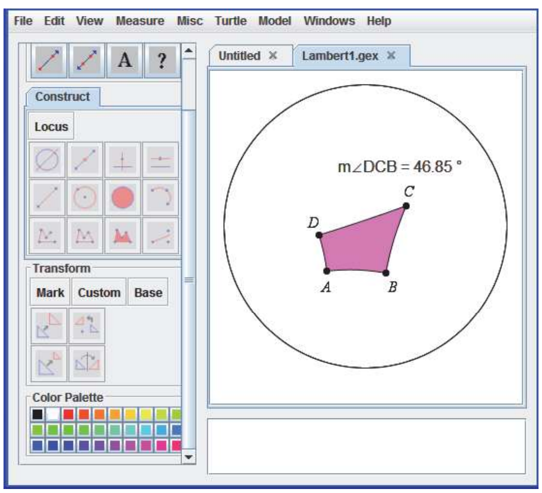
FIGURE 9
ANGLE C OF A LAMBERT QUADRILATERAL ABCD IS ACUTE



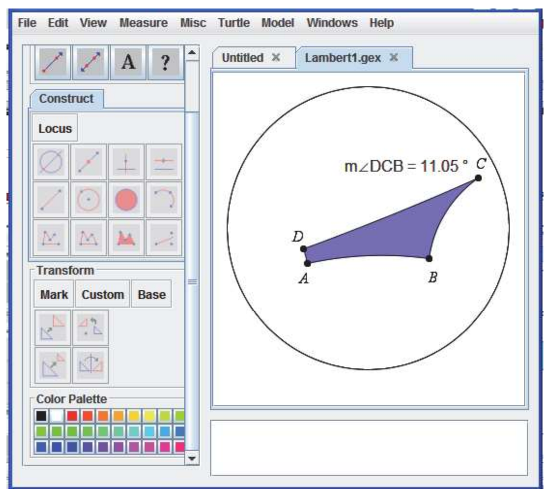
(a)



(b)



(c)



(d)

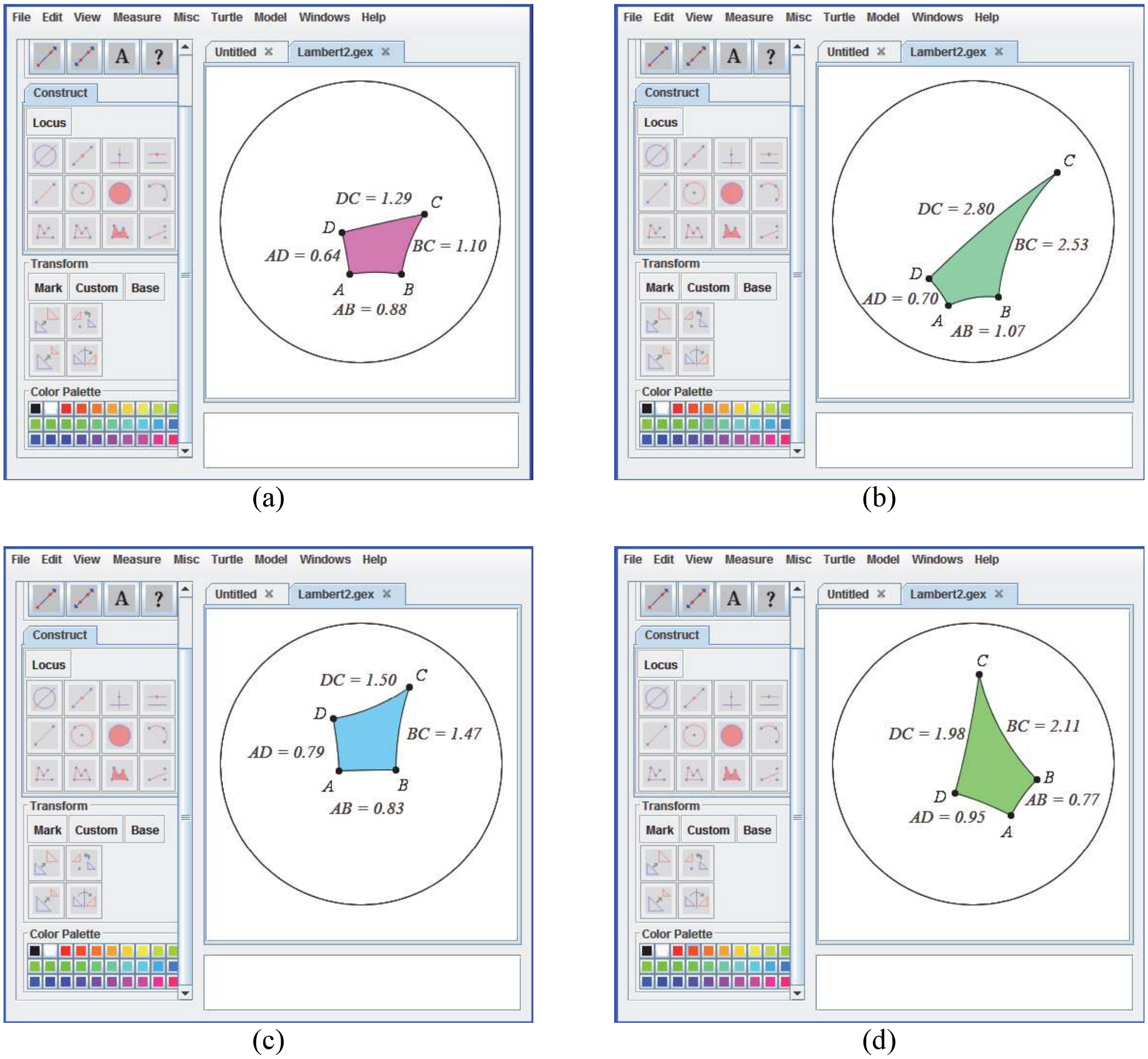
Exploration 5: In a Lambert Quadrilateral the Sides Adjoining the Fourth Angle (the One not Specified to be a Right Angel) are Greater than the Opposite Sides.

The researcher asked the students to draw another Lambert quadrilateral *ABCD* with right angles at $\angle A$, $\angle B$, and $\angle D$. Then, the students were asked to measure the sides of their Lambert quadrilaterals. They were asked to compare the lengths of the opposite sides in the Lambert quadrilateral *ABCD*, and answer the following questions: Are the opposite sides equal? In a pair of opposite sides can you characterize the one which is shorter?

After their observations on the numerical values (Figure 10(a), (b), (c), and (d), most of the students attained the following conjecture:

'The sides adjoining the acute angle of a Lambert quadrilateral are greater than the opposite sides.'

FIGURE 10
THE SIDES ADJOINING THE FOURTH ANGLE OF A LAMBERT QUADRILATERAL ARE GREATER THAN THE OPPOSITE SIDES



CONCLUSIONS

In this study, 40 preservice mathematics teachers explored important properties of Saccheri and Lambert quadrilaterals in the Poincaré disk model of hyperbolic geometry using the Dynamic Geometry program, the *Geometry Explorer*. Each of the preservice mathematics teachers individually formed their own examples on computer and compared each other results. Interestingly, they obtained the same results by different examples. Moreover, they had the opportunity to see the different examples of each other since they all formed different ones. The *Geometry Explorer* allowed preservice mathematics teachers to quickly and easily generate conjectures in hyperbolic geometry. They made conjectures and verify properties of the Saccheri and Lambert quadrilateral correctly and rapidly. They concluded that the summit angles of a Saccheri quadrilateral are always acute and equal and the segment joining the midpoints of the base and summit of a Saccheri quadrilateral makes right angles with the base and summit (is perpendicular to both). They also concluded that the fourth angle (the one not specified to be a right angel) of a Lambert quadrilateral is acute and the sides adjoining this acute angle are greater than the opposite sides. These properties were continuously discussed for centuries by Saccheri, Lambert and other geometers. While they can easily be explored within *Geometry Explorer* while dragging and exploring, these properties are certainly out of reach in traditional paper and pencil geometry. As mentioned by Straesser (2002), DGS–use widens the range of possible activities, provides an access route to deeper reflection and more refined exploration and heuristics than in paper and pencil geometry. Above all, the preservice mathematics teachers who knew no other geometry other than Euclidean geometry became aware of the existence of other geometries, hyperbolic geometry. Even though *Geometry Explorer* has changed the classroom environment into a more energetic and dynamic place, this does not mean that we support replacing the use of other instructional aids in the classroom with *Geometry Explorer*. It must be emphasized that the only way to make any geometry real and have a deeper understanding about the basic geometric concepts is to have students physically touch, play with, and do construction in the geometry.

Some properties of Saccheri and Lambert quadrilaterals in hyperbolic geometry need to be further investigated:

- Which is longer, the base or the summit of a Saccheri quadrilateral?
- Are the diagonals of a Saccheri quadrilateral congruent?
- Is the segment joining the midpoints of the sides of a Saccheri quadrilateral perpendicular to the sides? Which is longer, the segment joining their midpoints or each arm?
- Are the diagonals of a Lambert quadrilateral congruent?
- Is a Saccheri quadrilateral parallelogram?
- Is a Lambert quadrilateral parallelogram?

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