Provocations in Mathematics: Teachers' Attitudes

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This study analyses school mathematics teachers' attitudes towards using provocative mathematics questions in teaching and assessment as a potential pedagogic innovation. By a provocative mathematics question, we mean here a question designed to deliberately mislead the solver. It normally calls for an impossible task. For example, the question might ask for a proof of something that is not provable or show the existence of a solution of an equation that does not have a solution. Often a catch is based on a restricted domain or indirectly prompts the use of a rule, formula, or theorem that is inapplicable due to their conditions/constraints. Five groups of school mathematics teachers did a mini-test consisting of provocative questions. A post-test questionnaire was given to the teachers to obtain their feedback on the possible use of provocative questions in their teaching practice to enhance students' critical thinking skills. Teachers' responses are discussed and analysed in the paper.

Keywords: critical thinking, mathematics, teachers' attitudes

INTRODUCTION

Everyone can make and face mistakes in their life. The reality is that mistakes, errors, incorrect statements and misinformation are everywhere. Some of them are unintentional but some are made deliberately in order to mislead, misguide, and misinform like for example fake news. Regardless of the nature of a mistake, an ability to recognise it and act accordingly is a valuable skill. School mathematics curricula in all countries of which we are aware require the development of students' mathematical ways of thinking, in particular critical/analytical thinking as a learning outcome. Attention is also paid to developing students' confidence in mathematics. For example, the Finish Mathematics Curriculum specifically states that the students in the last 4 grades (6-9) should "learn how to trust themselves in mathematics" (Finnish National Board of Education, 2004, p.161). The Mathematics Curriculum in India "is explicit in mentioning the importance of encouraging children to freely express thoughts and emotions; this is seen to be a way to avoid copying what the teachers say. Children should develop their own voice". (Sumpter, 2015, p.127). Critical thinking is beneficial not only in the academic environment but also in other areas of life. Developing a sceptical habit of mind to analyse a mathematics question in a classroom should enhance an ability to analyse critically other situations outside mathematics. This is an important component of the theory of formal discipline, which asserts that mathematical understanding can be valuably transferred to other situations, see Inglis & Attridge (2017). It includes an ability to spot a mistake and avoid becoming the victim of mistakes.

Examples of Mathematical Mistakes

Mathematics is not immune from mistakes. Mistakes are in mathematics textbooks and dictionaries published by reputable publishers, highly ranked international research journals on mathematics education and national school exams. The four examples below would frustrate many mathematics teachers and lecturers.

Example 1. The textbook on engineering mathematics by Bolton (1997): "With a continuous function, i.e. a function which has values of y which smoothly and continuously change as x changes for all values of x, that we have derivatives for all values of x." (p. 332).

Example 2. The Collins Reference Dictionary of Mathematics by Borowski & Borwein (1989): "The function $y = \sqrt{(x+2)}$ whose graph is shown Fig.102…" (p.132).

FIGURE 1
COPY OF FIGURE 102 FROM EXAMPLE 2

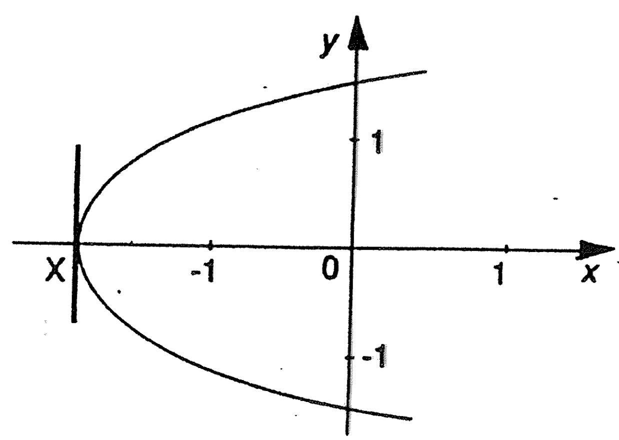
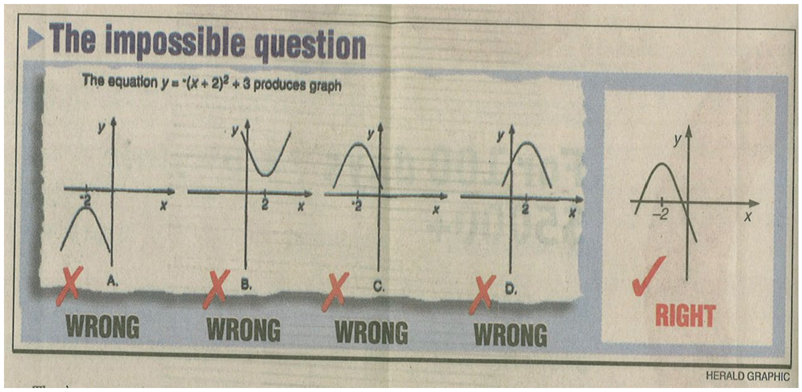


Fig. 102. **Critical point**. X is a critical point of the function.

Example 3. The journal article published in a top international journal in mathematics education (Saenz-Ludlow & Walganuth, 1998): "For both equations $x^2 + 2x + 1 = 0$ and $\sin^2 x = 1$ the equal symbol means equality for only 2 values of x". (p.155).

Example 4. The national school mathematics exam in New Zealand in 2000 where all four choices provided in a multi-choice question asking to match the correct graph of the quadratic function $y = -(x+2)^2 + 3$ were wrong. That led to a series of discussions nationwide including an article in the leading newspaper "The New Zealand Herald". The extract from the article is in figure 2.

FIGURE 2
THE EXTRACT FROM THE NEWSPAPER IN EXAMPLE 4



There are human inconsistencies even in typesetting mathematics, which could lead to error. Fateman & Caspi (1999) provided many examples of ambiguous use of writing mathematical expressions. One example is below:

Example 5. "We can argue that $1/2\pi$ means $1/(2\pi)$ because if we had meant $\pi/2$ we would have written it that way. Indeed, such hackishness has been encoded deliberately in Scientific Workplace, which interprets $\sin \pi/2$ as sin(pi/2) but the more general sin a/(b+c) as $\frac{\sin a}{b+c}$. There is an opportunity to ask the system to explain unambiguously what its interpretation is in such cases, but in our experience it is in human nature to fail to check such matters when they really count." (p.10).

The computer algebra community have been key in trying to avoid mistakes. Interesting examples are given in (Stoutemyer, 2011): "Most publications containing the Cardano solution of a cubic equation do not mention that his formula is not always correct for non-real coefficients. Consequently this formula has been misused by many people, including some computer algebra implementers, such as me. The consequences can be disastrous." (p.865).

We all notice mathematical mistakes from time to time in our everyday life, in particular in the media. Collections of mathematical mistakes can be a good resource for teachers like, for example the Australian project "Numeracy in the News" (https://www.tas-education.org/numeracy/). Here is one example from this resource: "…the case of an opposition member of the Tasmanian parliament who was outraged by the supposed drop of 500% in the tax paid by the local casino on its poker machine profits" (Watson, 2008, p.6). Another good resource is MisMaths, "a collection of mistakes, misconceptions and misrepresentations involving mathematics, which have appeared in some media or other, at some time" (http://www.cleavebooks.co.uk/trol/mismaths/trolmm.htm). Here is an example from that resource: "The ship is facing the wrong way and will need a 360° turn before sliding through the relatively tiny harbour entrance". One of the most famous books presenting a big collection of incorrect and misleading graphs published in books, magazines and newspapers is "How to Lie with Statistics" by Huff (1954).

Unfortunately, not all mathematical mistakes are just funny slips. Some of them led to tragedies. The recently published book "Humble Pi: A Comedy of Maths Errors" (Parker, 2019) in spite on its light subtitle presents many tragic real cases when mathematical mistakes caused the death of people. One such chilling story happened in 1986: "When the space shuttle *Challenger* exploded shortly after launch on 28 January 1986, killing all seven people onboard, a Presidential Commission was formed to investigate the disaster. As well as including Neil Armstrong and Sally Ride (the first American woman in space), the commission also featured Nobel prize-winning physicist Richard Feynman… The *Challenger* exploded because of a leak from one of the solid rocket boosters… The performance of the rubber O-rings was definitely the primary cause of the accident and remains the headline finding that most people remember… But Feynman also uncovered a second problem with the seals between the booster sections, a subtle mathematical effect... Checking if a cross-section of a cylinder is still circular is not that easy. For the boosters, the procedure for doing this was to measure the diameter in three different places and make sure that all three were equal. But Feynman realized that this was not sufficient… As well as the O-ring findings, and recommendations for how NASA should handle communication between the engineers and management, there is Finding #5: 'significant out-of-round conditions existed between the two segments'. NASA undone by simple geometry." (Parker, 2019, pp.75-77). The simple geometry is the fact that the converse of the statement "if we have a circle then the diameter is constant" is false, for example the Reuleaux triangle. However, the NASA engineers used the wrong criterion of a constant diameter to identify a circular shape. That mathematical error contributed to the disaster as reported in Finding #5 of the Investigation Report quoted above.

How to Deal With Mathematical Mistakes

In spite of the clear evidence of existence of mathematics mistakes and misuses of mathematics, there is scant mathematics education research conducted on how to deal with this issue. One obvious way is to improve the quality of published resources on mathematics. At the very least, mathematics teachers should carefully select teaching materials (textbooks, websites, articles, etc.) before accepting and recommending them as teaching/learning resources for their students. Some countries take it seriously and impose fines on publishers that publish textbooks with mistakes: "In a recent review of textbooks proposed for adoption in California, a panel of mathematicians found hundreds of errors. While the mistakes ranged from a missing equals sign to a muddy explanation of the quadratic equation, it was their pervasiveness that surprised state officials most…"It was shocking," said Cathy Barkett, the administrator of the curriculum-frameworks and instructional-resources office. "In one 200-page text, 50 of the pages had errors."…Hefty fines will be levied for mistakes that publishers had previously agreed to correct--up to $25,000 and 1 percent of the sales for major errors that impede student learning, and $5,000 for less serious ones". (Manzo, 1999, pp.6-7). Kajander and Lovric (2009) investigated the role of mistakes in mathematics textbooks in developing students' misconceptions. They conclude: "Situations leading to potential misconceptions occurred consistently in multiple sources. Acknowledging that textbooks remain a fundamental teaching resource, we suggest that more attention be paid to the presentation of mathematics. Furthermore, analyses of textbooks should include developmental as well as subject matter scrutiny" (p.180). Another way to deal with mathematics mistakes is try to prevent them. Sangwin (2015) claims that "many of the classic *problems, fallacies* or examples of *false reasoning* can be avoided by using an *audited elementary algebra*. The word *audited* refers to the explicit tracking of domains throughout the calculation. Auditing (i) eliminates many of the problems of spurious or missing solutions when solving equations, (ii) reveals where in a chain of reasoning such problems occur, and (iii) is a natural extension of algebra which does not introduce any artificial devices" (p.298). Cipra (2000) suggests a number of strategies and a series of exercises in calculus to look for possible mistakes and "how to find them before the teacher does". He argues: "Everybody makes mistakes. Young or old, smart or dumb, student or teacher, we all make 'em. The difference is, smart people try to *catch* their mistakes" (p.xi).

A common way to prepare students to deal with mathematical mistakes is asking them to disprove incorrect mathematical statements by counterexamples. In an international study (Gruenwald & Klymchuk, 2003) involving more than 600 students from 10 universities in different countries, 92% of the participating students found the use of counterexamples to be very effective. The students reported that it helped them to understand concepts better, prevent mistakes, develop logical and critical thinking, and made learning mathematics more challenging, interesting, and creative. A practical experience of using counterexamples as a pedagogical strategy in an introductory calculus class was shared in Klymchuk (2014). Some of the strategies include: "On different occasions, I have given students mixtures of correct and incorrect statements, asked students to create their own wrong statements and associated counterexamples, made deliberate errors in my lecture and moved on with the hope that students would detect them, asked students to spot errors in their textbook, given students extra credit for providing counterexamples to challenging statements I posed in class, and included on assignments and tests questions that require students to construct counterexamples" (p.1264). A collection of incorrect statements in an introductory calculus and suggested counterexamples illustrated by graphs can be found in Klymchuk (2010). Abramovitz, Berezina and Berman (2003) shared their similar practical experience of giving students incorrect proofs and asking them to identify mistakes (by a definition it is a *sophism* although the authors did not use this word). They also used such tasks in tests and exams. The feedback from their students was very favourable and the authors concluded: “It was found that explaining the mistakes is an effective way to improve students’ understanding of the theory. Examinations are an essential part of the learning process, a part that is highly respected by students. Motivated by these remarks we discussed in this paper several examples of wrong proofs used to help the students to understand some fine points in calculus and improve their understanding of the subject.” (p.764).

An innovative pedagogical strategy to prepare students and teachers to deal with mathematical mistakes and enhance their critical thinking was suggested in Klymchuk (2015). The idea was to use so-called provocative questions. A provocative question is a non-routine mathematics question that looks like a simple routine task but in fact, it has some catch. It normally calls for an impossible task. For example, the question might ask for a proof of something that is not provable; or show the existence of a solution of an equation that does not have a solution; or find a certain feature or characteristic (e.g. area, derivative) of a mathematical object (e.g. triangle, function) in a case where the object itself does not exist. Often a catch is based on a restricted domain or indirectly prompts the use of a rule, formula, or theorem that is inapplicable due to their conditions/constraints. A provocative question often provokes a solver to find a wrong solution. The intention of using such questions is to test solver’s attention, critical thinking and confidence in a hope that they would find and avoid the catch. Provocative questions as understood in this paper closely relate to recreational mathematical puzzles. Probably the first provocative mathematics question in the Western history was posed by Alcuin of York, an English scholar born around 732 AD, in his book “Problems to Sharpen the Young”. An annotated translation of his book *Propositiones ad acuendos juvenes*, the oldest mathematical problem collection in Latin, is given in Harley and Singmaster (1992). One of the 56 problems presented in the book asks for an impossible task:

> “Problem 43. A man has 300 pigs and orders that they are to be killed in 3 days, an odd number each day. (There is a similar puzzle with 30 pigs). What odd number of pigs, either of 300 or of 30, must be killed each day?” (p.121).

Provocative questions can often be found in books on puzzles. An example of an impossible task from Badger et al. (2012) is below:

> “Task 8. There are two telephone poles, perpendicular to level ground. Each one is 30 m tall. The poles are an unknown distance apart. A 50 m cable is to be strung from the top of one pole to the top of the other. Because the cable is heavy, it will of course droop and take up the shape of a catenary. What must the distance between the two poles be so that the lowest point of the cable touches the ground?” (p. 6).

Many teachers use puzzles in a classroom as a fun break. However, specifically designed provocations tailored to the topic and used not only in a classroom but also in *assessment* is a pedagogical strategy we seek to investigate. This paper deals with school mathematics teachers’ attitudes towards this strategy.

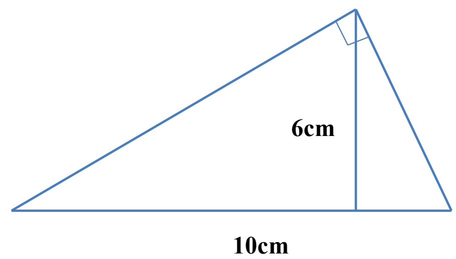
THE STUDY

At the first stage, a mini-test consisting of 7 provocative questions and a post-test questionnaire were given to four groups of school mathematics teachers from Germany, Hong Kong, New Zealand and Ukraine to investigate teachers’ level of attention (Klymchuk, 2015). Later, the same study was replicated in Australia and reported in Brown (2018). The total number of participants in all five groups was 127. In spite of the warning that some questions might contain a catch, the majority of the teachers solved most of the questions incorrectly.

Examples of the questions from the mini-test are below.

Question 1. Find the area of the right-angled triangle if its hypotenuse is 10 cm and the height dropped on the hypotenuse is 6 cm.

FIGURE 3
THE DIAGRAM FOR QUESTION 1



Solution: There is no sense to talk about the area, as the triangle does not exist. By the Thales' theorem, the hypotenuse in a right-angled triangle is a diameter of its semicircle so in this case the height cannot be bigger than 5 cm.

Question 4. Prove the identity $\sin x = \sqrt{(1-\cos^2 x)}$.

Solution: The 'identity' is not true. Squaring both sides does not prove it because this operation is irreversible. It is not an identity but an equation with infinitely many solutions $x \in [2\pi n, \pi(2n+1)]$.

Question 6. Find the derivative of the function $y = \ln(2\sin(3x) - 4)$.

Solution: The derivative does not exist because the function does not exist, as the argument of the log function is always negative.

The above two studies analysed teachers' feelings and reasons for their performance on the test using theories of selective, divided and focused attention from psychology and Mason's (2002) concept of discipline of noticing. This paper focuses on teachers' responses to the last question of the questionnaire, in particular on the use of provocative questions in assessment.
The last question of the questionnaire is below.

Question 3. Would you make any changes in your teaching practice after doing the test? If so – which changes? If not – why?

In both German and New Zealand groups (10 and 14 teachers respectively), all participants reported that they would make changes in their teaching practice after doing the test. Common comments were as follows: "Introduce tricks like this to class to make them think; keep encouraging and creating environment where a deep conceptual knowledge is cultivated; encourage and reward checking of answers; more emphasis on the validity of solutions; teach them to examine the question thoroughly; give students more questions that will force them to think about the conditions surrounding the questions; I would encourage students to think through questions carefully; I try to make my students think more about restricted domains, check solutions and not trust graphical calculators; give them problems occasionally that will 'trip' them up if they have not gone back and re-assessed their solutions; more emphasis on the nature of problem solving; stop answering impulsively, think before respond; I will expose students to such questions to get them to think more deeply about the conditions; re-think exercises; discuss more special cases;

implementing exercises with surprising answers; more emphasis on self-control". Only one teacher made the comment related to assessment: "Unless it is an element of the assessment I might not have time."

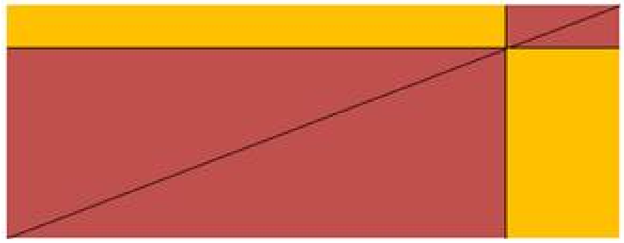
In the Hong Kong, Ukrainian and Australian groups (26, 26 and 51 teachers respectively), about half of the participants reported that they would make changes in their teaching practice with common comments as follows: "stimulate student's thinking, solving the questions not only according to an algorithm, be attentive rather than solving automatically, develop logical thinking, develop thoughtfulness and reasoning; it is very easy to follow a rule / algorithm / formula, but unless you have the understanding 'why' you cannot see when there may be no solution". The other half reported that the questions from the test were not common and not part of assessment and therefore they would not change their teaching practice. The most strong comments about rejecting provocative questions in teaching and assessment came from the Australian group: "In my teaching practice my students are given only problems which are possible to solve, they follow the script; at school we don't tend to pose impossible questions; exams always have questions which make sense, so why teach them beyond the process; I am worried that in an assessment they will become absorbed by looking for the trick and waste precious time, as the assessments they do, do not have trick questions; we are trying to get them to be successful in their WACE [West Australian Certificate of Education] exams after all".

DISCUSSION AND RECOMMENDATION

About 60% of the participants from both studies reported that they would make changes in their teaching practice by introducing 'trick' questions in their teaching, encouraging students to 'question the question' and analysing conditions and constraints before applying a certain formula or theorem. The remaining 40% of the participants probably tend to 'teach to the test' which consists mainly of procedural routine questions. They clearly indicated that they would not use provocative questions, as they are not in the assessment. Brown (2018) argues: "If senior secondary mathematics assessments contain few questions which explore full understanding, the questions presented to "trusting" students must be routine, the content pre-negotiated and expected. Students who are not exposed to the risk of losing their self-confidence are therefore not being challenged in assessments to demonstrate more than procedural competence. This suggests there may be a conspiracy between teachers and their students to avoid coverage of true conceptual understanding in senior school mathematics tests and examinations." (p.191). Klymchuk (2015) suggests: "Including the type of questions from the mini-test into the assessment would encourage those teachers to pay more attention to details and analysis and enhance such skills in their students. After all, many situations in real life do not have a single 'correct' answer as is the case with routine questions from traditional assessments in mathematics." (p.69).

It is easy to create or collect provocative questions at any level, use them in a classroom and include them into assessment. For example, in an introductory calculus course along with a routine procedural question like "Find the derivative of the function $y = \cos \ln x$" one can include the provocative question "Find the derivative of the function $y = \ln \ln \sin x$" that tests completely different skills (the Chain Rule is not applicable as the function has an empty domain). Or, along with a conceptual question like "Sketch a graph of the derivative of the function based on the graph of the function provided" one can include the question "Sketch a graph of a function that is differentiable on the interval (a,b) and discontinuous at least at one point on (a,b)" that tests completely different skills (it is impossible as a function differentiable on (a,b) is continuous on (a,b). At a lower level, one can include questions like these "Show that the area of the triangle with the sides 20 cm, 10 cm and 8 cm is larger than 50 cm^2" or "Explain why a longer number means a larger number" or "Prove that the orange rectangle on the right has a larger area than the orange rectangle on the left in figure 4".

FIGURE 4
THE DIAGRAM FOR THE QUESTION ON THE AREAS OF THE RECTANGLES



There is another cunning way to create a provocative question apart from asking someone to do an impossible task in a way that it looks like it is possible. The trick is to deliberately divert attention into a wrong direction of thinking. As an example, let us look at the following recreational puzzle.

Puzzle. Two coins total 30 cents. One of them is *not* a 10-cent coin. What are these coins?

Using italic font to emphasise the word 'not' the attention is deviated to the wrong direction of thinking such that there are no 10-cent coins. Surprisingly, many people are stuck with this simple puzzle as they replaced the condition 'one of them' with 'none of them' due to the shift of attention.

Another example of the diversion of attention is demonstrated by the following famous conjunction fallacy created by psychologists Tversky and Kahneman (1983).

"Linda Problem. Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Which is more probable:

a) Linda is a bank teller.
b) Linda is a bank teller and is active in the feminist movement."

Here the Linda's background provokes to the wrong thinking. In spite of the clear logical reasoning that the conjoint statement is less probable, between 85% and 90% of people participated in multiple experiments of Tversky and Kahneman gave the wrong answer (b).

There are well-known collections that can inspire the design of provocative questions, see for example: Maxwell (1959), Northrop (1945), Posamentier & Lehmann (2013). The website resources like Numeracy in the News and MisMaths mentioned earlier are also useful.

Typical assessment in mathematics consists of the following three types of questions/problems: procedural, conceptual and applications. In the vast majority of such questions, all conditions of theorems/formulas/rules are met. Therefore, students might develop a habit of applying formulas and rules without checking the conditions/constraints. In real life, not everything behaves so nicely and ignoring conditions/constraints might lead to significant and costly errors. The intention of including provocative questions in mathematics assessment is to prepare students for real life by enhancing their critical thinking that includes the abilities to analyse questions and recognize mistakes. These abilities can be transferred outside mathematics classroom into everyday life so students become better-informed citizens. As a pedagogical strategy, provocative mathematics questions are *deliberately* designed to mislead the solver. They demonstrate the importance of being alert and ready to analyse everything. They enhance a habit of questioning the question and not to take anything for granted which is an essential part of a mathematical way of thinking. We suggest including provocative questions as the fourth type of questions in any mathematics assessment along with the common three types - procedural, conceptual and applications. There might be a better and more positive word to describe this type of questions instead of 'provocative'. We suggest that the inclusion of provocative questions should be gradual: first use them as an additional, extracurricular activity; then include them in the mathematics curriculum and subsequently into formative assessment and later summative assessment. Practice in solving and posing provocative questions should be an integral part of training of prospective mathematics teachers, and is included into professional development of in-service school mathematics teachers. Taking into account a solid professional background of mathematics teachers, the investment in training them in using provocative questions in a classroom and assessment might be very small – attending just 1-2 seminars or workshops – however the benefits for their students and society is enormous. These benefits include but not limited to recognising mistakes and fake news, identifying contradictory information, eliminating impossible cases, using sceptical and unbiased analysis, making rational judgement and decisions based on factual evidence, and other traits of critical thinking.

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