Connecting Experimental Uncertainty to Calculus and to Engineering Design

Gregory Hill
University of Portland

Tamar More
University of Portland

Experimental uncertainty and the epsilon-delta proof provide a rich example of a thread that weaves together topics in introductory physics, calculus, and engineering design. This thread presents an opportunity for deepening the connection between calculus and physics, strengthening conceptual understanding in both fields, and expanding students' understanding of both experimental uncertainty and propagated error as constraints informing the design of an experiment rather than simply an unavoidable consequence of some procedure. We discuss the connections and describe several activities that help students explore the thread.

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INTRODUCTION

As part of an integrated calculus and physics course, we have developed a number of explorations of topics that connect the two disciplines. Here, we describe in detail one such example, a series of activities in which we address the connection between experimental uncertainty and the ε - δ proof, and list a few other examples in brief. This exploration provides such an opportunity for deepening the connection between calculus and physics, while motivating and strengthening conceptual understanding in both fields. Physics students often have a sketchy understanding of experimental uncertainty and of the formulas for propagated errors. Rarely do they feel that they can control the experimental uncertainty. For these reasons, students often misuse the formulas or fail to take the uncertainty seriously. Similarly, calculus students struggle to gain an intuitive grasp of concepts such as the ε - δ definition of the limit and the concept and application of differentials.

In a typical traditional physics course, the discussion of experimental uncertainty in a measurement or calculation is primarily focused on a means for comparing results or judging the meaningfulness of a result. Students tend to interpret the uncertainty even more narrowly, only as a means for comparing their own measurement against some quantity that they believe to be known exactly. While all of these are indeed important in experimental work, they do not tell the whole story. They rarely offer students a deeper insight into their work, and even more rarely provide an opportunity for further predictions and tests. Furthermore, the students' tendency to use the uncertainty as an afterthought, to check for agreement between their answer and the "expected" one, diminishes the motivation for doing the experiments in the first place. After all, if they already know the answer, are they not justified in viewing the lab as busy work? We want our students to see both that experimental uncertainty plays an essential role in the very design of experiments and that they do, in fact, have some control over the uncertainty. Often the experimenter must ask the question: "to what accuracy do I need to measure or control various experimental parameters in order to achieve my desired accuracy in another measured or calculated quantity?" This question is quite practical, and the idea that uncertainty is a fundamental, yet tractable aspect of experiment is inherent in it. Instead of being retrospective, it provides an obvious predictive problem: "How can I design (or, even, can I design) an experiment that achieves a given accuracy, does my design work in practice, and what tradeoffs would it require?"

From the mathematical viewpoint, this application of uncertainty to experimental design has an analog that is quite timely for the introductory student. In typical calculus texts the application of differentials to error propagation is largely limited to calculating the error in $y=f(x)$ given some uncertainty in x. The often dreaded ε - δ definition of the limit, in which the limit of a function is given a precise definition (Stewart, 2003), addresses the same question from a mathematical perspective: What range in the argument of a function (the controlled parameter) corresponds to a specified range in the value of the function (the calculated result)? These two topics typically occur early in both the physics and calculus curricula, so by using our methods we can enhance the perceived value of each subject for the other. In the remainder of this paper, we present some hands-on student investigations of the mathematics, the physics, and the connection between them.

THE UNCERTAINTY THREAD IN AN INTEGRATED COURSE

As in many standard introductory physics labs, early in the first semester our students are introduced to measurement error by timing a falling ball and using the data to calculate the distance the ball has fallen. Since the quantity measured (time) is not necessarily the quantity desired (distance), students see that if they are to take experimental uncertainty seriously, they have to deal with the error in a calculated value. We use this physics experiment to motivate and reinforce material in the calculus portion of our course in several ways.

You may recall the following technical definition of the limit of a function from your calculus class:

$$\lim_{x \to a} f(x) = L \tag{1}$$

if for every number ε > 0 there is a number δ > 0 such that

$$|f(x) - L| < \varepsilon \quad \text{whenver} \quad 0 < |x - a| < \delta$$

A calculus instructor will describe the role of ε as setting a required closeness of $f(x)$ to L which can be achieved by requiring x to be within δ units of a. Students typically regard this as quite abstract and, in fact, the ε - δ definition of the limit is often skipped in calculus courses. To give this definition concrete meaning we employ an analogy using experimental error. In our interpretation, x is a measured quantity (time in the experiment above,) $f(x)$ is a value calculated using this measurement (height of the falling ball in this case), a plays the role of the "true" value of the measured quantity, and L is the "true" value of the calculated quantity. The difference $|x\text{-}a|$ is thus the experimental error while $|f(x) - L|$ is the uncertainty in the calculated value. In this interpretation, ε is a tolerance for the error in the calculated value, while δ is the bound on the error in the measured quantity required to obtain a result consistent this tolerance. Calculus students typically regard the activity of applying the ε-δ definition as a meaningless game in which they are given ε and asked to find the corresponding δ that makes the formula "work". With our physics interpretation, this "ε-δ game" becomes the fundamental question "how accurately do I have to measure x to get a desired accuracy in the calculated value $y=f(x)$?" The standard example of $y=x^2$ used in every calculus book now has added meaning in the context of a measured time t being used to calculate a desired distance $y= 1/2\ g$

t^2 (with known acceleration due to gravity *g*.) The same mathematics also applies when students measure height and velocity for a falling ball and demonstrate conservation of mechanical energy.

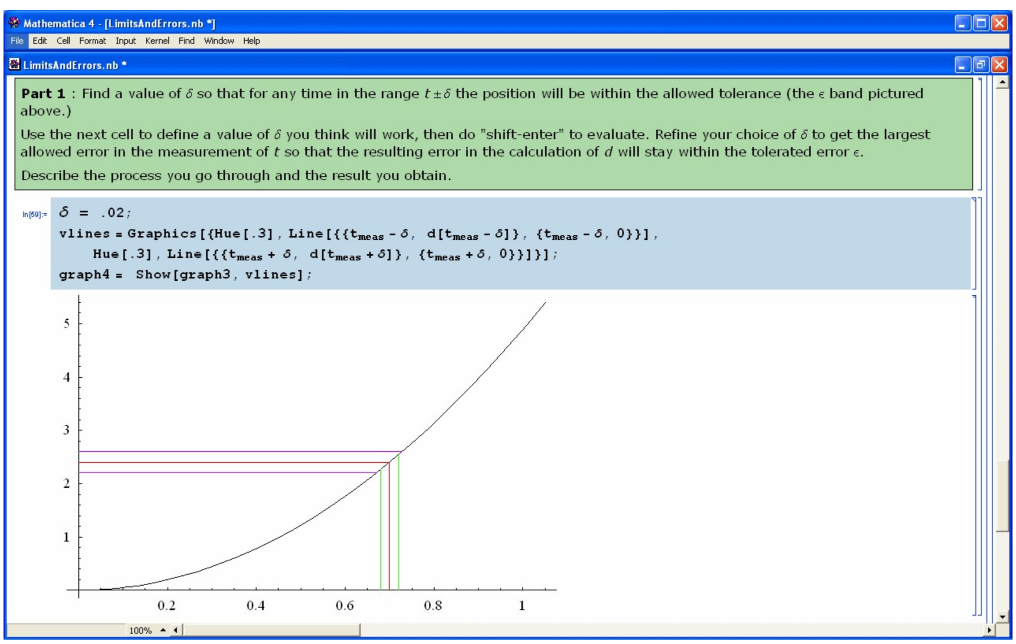
ACTIVITIES

Our activities continue with mathematical experimentation based on the physics experiment of measuring time and distance for a falling ball. The function in question is $y(t) = 1/2\ g\ t^2$ and we are examining a limit of the form

$$\lim_{t \to a}\left(\frac{1}{2}gt^2\right) = L \tag{2}$$

where *a* is a fixed value of *t*. The statement of the problem in the lab reads "Find a value of δ (the constraint on the uncertainty of the measurement) that will guarantee an error of no more than ε in the calculated distance *y(t)*." Given various values of ε, student use the mathematical programming package Mathematica to find, using trial and error, the values of δ that correspond to these ε's. They then plot the results to discover empirically that the graph of δ vs. ε forms a line (for small values of ε) whose depends on time in a simple way. Examples of the output of the notebook and the resulting graphs are shown in figure 1. Having discovered that the equation $\varepsilon = gt\delta$ describes their results for the trial function, they often notice that the slope of the line is the derivative of the function. Students are then motivated to ask why that relationship holds, and we carry out analytic calculation to derive this equation relating ε and δ.

FIGURE 1
MATHEMATICA NOTEBOOK: FINDING Δ FROM E USING TRIAL AND ERROR



Now that we have developed the mathematics in the context of allowed uncertainties, we can apply the ideas in an experimental physics context: suppose we would like to find an unknown height by measuring the time it takes for a ball to fall that height. How can we design an experiment to find the height with an uncertainty of 5 cm? Is it even practical to try to measure it to 1 cm using this method? We translate this problem to an ε-δ question: We call the allowed error in the height ε and the time uncertainty δ. By making a very rough estimate of the unknown height, we can calculate the approximate expected time and find δ using the results from the Mathematica activity.

This calculation naturally raises the practical experimental design question: given the available equipment and time, how do we actually achieve the δ that we calculate in this experiment? One obvious answer is that the uncertainty in the average time is a function of the both the accuracy of each timing measurement and the number of measurements. Following a standard activity in Workshop Physics (Laws, 1997), we first drop the ball 20 times from a known height. The standard deviation of the timing measurements tells us the uncertainty in each timing measurement, and we define the statistical uncertainty as twice the standard deviation of the mean (SDM.) By varying the number of measurements, we can achieve any desired accuracy (δ) in the average time.

In order to underscore the meaning of statistical uncertainty, we divide the class into four groups. Each carries out the experiment. Rather than compare the results to a "known" height, measured in some other, presumably better way, we compare the results of the various groups. Typically, the range of the average times does correspond fairly well to the experimental uncertainty as defined above. To illustrate the difference between random and systematic error, we then measure the height with a ruler, a measurement that usually does not agree, within the uncertainty, with the calculated distances. This, of course, is due to systematic error, most likely introduced by the reaction times of the experimenters.

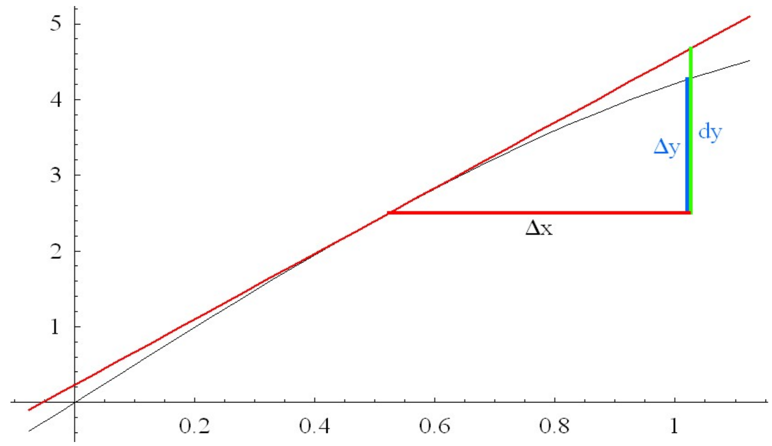
In our experience, the statistical uncertainty (as opposed to the difference from the tape-measured value) in the calculated heights has been about 0.1m. The calculus activity suggested that if we want to decrease the uncertainty in the calculated height by a factor of two, we would have to decrease the uncertainty of the average time by a factor of two as well, which requires four (2^2) times as many trials. Looking back at our question "Can we even carry out an experiment to measure it to 10 mm?" the answer becomes obvious: this would require in increase by a factor of 100 in the number of measurements, which is clearly impractical. Even taking two sets of 80 measurements (to achieve reduce the uncertainty by a factor of two) can be quite time consuming, but here we can combine all the measurements of the entire class to get one large set. We need only repeat the experiment once and pool the new measurements to get a second large set. Students are often surprised to see that the new averages of the larger data sets are much closer to each other, and the calculated heights do in fact usually agree with each other within the predicted 5cm. Since the systematic error is still present, the disagreement with the "theoretical" ruler-measured value becomes even more apparent.

THREAD CONTINUATION: DIFFERENTIALS

Later in the physics course, the students encounter another situation in which the uncertainty in a calculated value can be of crucial importance: finding the range of a projectile in terms of the initial launch angle θ, and, in the spirit of our approach, deciding how accurately we need to aim in order to hit the target.

The earlier analytic calculation, in which we derived a formula for δ in terms of ε, is too difficult in this case. This physics problem motivates the calculus segment of the course to develop the technique of differentials as a way to estimate the relationship between ε and δ. After reminding students of the observation they made in the earlier lab ("just divide by the slope!") the material on differentials makes sense and a natural extension of the earlier investigations. In a calculus lab, the students investigate the relationship among the three quantities picture in figure 2: the uncertainty in x (Δx), the uncertainty in y (Δy), and the estimation of the uncertainty in y (dy) found using the equation $dy = f'(x)\ \Delta x$.

FIGURE 2
MATHEMATICA NOTEBOOK: SHOWING THE RELATIONSHIP BETWEEN DIFFERENTIALS AND UNCERTAINTIES. THE LOWER CURVE IS Y = F(X) WHILE THE UPPER LINE IS A TANGENT LINE



The discussion of differentials is presented in the calculus course in terms of the standard notation of mathematics, but with explicit correspondences made with the physics language. As in the case of the ε and δ material, physics and mathematics often develop the same ideas using different languages. One great advantage of integrating or at least coordinating the classes is the opportunity to address these differences directly, essentially developing a math/physics dictionary. This linguistic issue may seem trivial, but is often a crucial barrier to students' ability to apply mathematics techniques in a physics context or to understand a mathematical idea using physical models.

DISCUSSION AND ADDITIONAL APPLICATIONS

Admittedly, the immediate student response to the limits and uncertainty thread, and to the integration of the math and physics in general, has been somewhat mixed. From interviews with students, we find that the coordination and integration of the math and physics curricula does provide them with a better sense of the connection between the subjects and that the activities keep them engaged. Many students do appreciate both having the math they need in time to use in the physics course as well as seeing the applicability of the math concepts. They also appreciate the active component of the course. The following are some representative comments from interviews and evaluations: “The lab really helps enforce learning the whole concept” (interview with a student.) ``This course is good because it coincides with the math class that is relevant for what we are learning in physics, which makes it easy'' (physics course evaluation.) “I liked how there was a strong focus on using the calculus to describe the physical world, cementing the correlation between math and physics” (another physics course evaluation.) “From taking both individual calculus and physics classes, I was quite confused with the calculus based physics until we learned it in calculus. I told students that this class may be tougher, but you will know more than other students” (interview with a peer leader.)

We have, however, found that some students prefer to think within small, well-defined boxes and will resist applying the connections we are encouraging them to make. The activity we have described presents an example of this tendency, even among some of the students who profess an appreciation of the integration and the active learning component. In the calculus ε-δ activity, the problem is explicitly given in a physics context. The physics experiment in which they estimate the number of timing measurements needed to achieve a certain accuracy takes place the day immediately after the math class ε-δ activity. The students are reminded of this activity and their work on it, and told to use their results to estimate the number of trials. Still, when asked how they chose this number, the vast majority reported they had guessed it, rather than use the activity. While this supports our initial sense that even students who are proficient at calculating uncertainties in the traditional context do not necessarily have the deeper understanding of their role in experimental design nor the connection to the mathematics, it suggests that even the activity we describe is insufficient by itself, and a second (or third) visit to the topic is in order. Experimental verification of the work energy theorem and conservation of mechanical energy are two obvious candidates for this within the same course. In the later electricity and magnetism course, additional opportunities for using uncertainty in experimental design include determining charge from the angle of an electroscope deflection, capacitance from an RC charging graph, or the electron charge to mass ratio from the radius of its circular path and applied magnetic field.

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