Remarks on Fixed Point Approaches to Insurance and Finance

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Fixed point theory has been applied to various practical problems of insurance and finance during several decades. In our earlier paper (Voutilainen, 2022) we have presented problem classes tackled in the literature by fixed point methods. In Voutilainen (2023) we study fixed point problem solution methods with the help of solutions of equilibrium problems in several classes. For this paper we have gathered and commented on some specific fixed point applications to the insurance and finance areas. Many of them also adopt other interesting theoretical areas. It turns out that fixed point theory really has a number of important applications both to insurance and finance and to theoretical mathematics.

Keywords: fixed point approaches, fixed point theory, insurance, finance

INTRODUCTION

Fixed point theory has been applied to various practical problems of insurance and finance for several decades. In our earlier paper (Voutilainen, 2022) we have presented problem classes which have been tackled in the literature by fixed point methods. In Voutilainen (2023) we study fixed point problem solution methods with the help of equilibrium problem solutions. For this paper, we have gathered and commented on specific fixed point applications to the insurance and finance areas. Many of them also adopt other interesting theoretical areas.

In chapter 2 we discuss fixed point approaches to insurance, and in chapter 3 approaches to banking and finance. In Chapter 4 we highlight some recent theoretical developments.

INSURANCE APPROACHES

In Voutilainen (2022) we review insurance-related equilibrium research found in the literature like insurance market equilibrium, reinsurance, insurance markets with asymmetric information, adverse selection, monopolistic/competitive insurance markets, deposit insurance, moral hazard, Internet, signalling, reputation, linear equilibria and Nash equilibria. Equilibrium is closely connected with fixed points, and proofs of several equilibrium theorems use fixed point theory as Voutilainen (2022) points out. In this paper we review direct fixed point approaches to insurance and finance.

The concept of ruin is central in classical risk theory. Ruin is the most serious situation for an insurance company because it is followed by aborting or terminating insurance business. It is therefore no surprise that ruin related aspects are widely discussed in literature.

The paper by Gajek (2005) presents an algorithm of bounding from above and from below the deficit distribution at ruin. The algorithm is based on iterating a monotone integral operator which has a fixed point at the deficit distribution at ruin. The upper and lower bounds converge monotonically with an exponential rate to the exact value of the deficit distribution. A counterpart of the famous Cramér-Lundberg inequality for distributing the deficit at ruin is established.

Avram et al. (2016) consider a one-dimensional surplus process with a certain Sparre Andersen-type dependence structure under general interclaim times distribution and correlated phase-type claim sizes. The Laplace transform of the time to ruin under such a model is obtained as the solution of a fixed-point problem, under both the zero-delayed and the delayed cases. An efficient algorithm for solving the fixed-point problem is derived together with bounds that illustrate the quality of the approximation. A two-dimensional risk model is analyzed under a bailout-type strategy with both fixed and variable costs.

Gajek and Rudz (2018) apply Banach Contraction Principle (see Banach fixed point theorem in Voutilainen, 2023) to approximate a vector Ψ of ruin probabilities in regime-switching models. A Markov chain is interpreted as a ‘switch’ that changes claims’ amount and/or wait time distributions. The insurer has a possibility to adapt the premium rates in response. An associated risk operator L is proven to be a contraction on a properly chosen complete metric space while Ψ is shown to be the unique fixed point of L within this space. Thus, by iterating L on any of its points, one can simultaneously approximate Ψ and control the error of approximation. Numerical examples confirm high accuracy of the resulting procedure.

Jiang (2019) studies optimal dividend distribution for an insurance company whose risk reserves in the absence of dividends follow a Markov-modulated jump–diffusion process with a completely monotone jump density where a finite-state irreducible Markov chain modulates jump densities and parameters including discount rate. The major goal is to maximize the expected cumulative discounted dividend payments until ruin time when risk reserve is less than or equal to zero for the first time. Jiang extends his earlier results for a Markov-modulated jump–diffusion process from exponential jump densities to completely monotone jump densities by proving that it is also optimal to take a modulated barrier strategy at some positive regime-dependent levels and that value function as the fixed point of a contraction is explicitly characterized.

Jiang (2022) supposes that risk reserves of an insurance company are governed by a Markov-modulated classical risk model with parameters modulated by a finite-state irreducible Markov chain. This paper’s main purpose is to calculate the ultimate ruin probability that ruin time, the first time when risk reserve is negative, is finite. Jiang applies Banach contraction principle, q-scale functions and Markov property to prove that ultimate ruin probability is the fixed point of a contraction mapping in terms of q-scale functions and that ultimate ruin probability can be calculated by constructing an iterative algorithm to approximate the fixed point. Unlike Gajek and Rudź (2018), Jiang’s paper uses *q*-scale functions to obtain more explicit Lipschitz constant in Banach contraction principle in his case so that proofs of several Lemmas and theorems in their Appendix are unnecessary and some of their assumptions are confirmed in his case.

The purpose of the Zhong and Huang (2023) paper is to explore a discrete-time cash flow optimization problem of the insurance company with time value of ruin under different interest rates. To consider the time value of ruin, they assume that the shareholders can get subsidies per unit time, as long as the insurance company is not bankrupt. A stationary Markov chain controls the switching of different interest rates on the market. The dynamic programming principle is used to solve this optimization problem. By using a method of fixed-point theory, they show that the value function is the unique solution of the dynamic programming equation and a numerical algorithm is proposed to solve the value function and the optimal policy.

Liu et al. (2023) investigate ultimate ruin probability, the probability that ruin time is finite, for an insurance company whose risk reserves follow a Markov-modulated jump-diffusion risk model. They use the Banach contraction principle (fixed point theorem) and *q*-scale functions to prove that ultimate ruin probability is the only fixed point of a contraction mapping and show that an iterative equation can be employed to calculate ultimate ruin probability by an iterative algorithm of approximating the fixed point. Using *q*-scale functions and the methodology from Gajek and Rudź (2018) applied to the Markov-modulated jump-diffusion risk model, Liu et al. get a more explicit Lipschitz constant in the Banach contraction principle. Dynamics of insurance markets are discussed in the next two papers.

Siemering (2021) investigates the dynamics of an insurance market in which insurance companies may dishonestly deny eligible claims. Behaving dishonestly can increase the current profit but also entails the risk of losing profit in the future due to a worse reputation. Depending on the reputation cost imposed by policyholders, the analysis either predicts the emergence of reputation cycles or convergence to a stable equilibrium in which all eligible claims are accepted and the insurers' reputations remain high. There is convergence to a fixed point where insurers are disciplined not to behave dishonestly and policyholders have trust in the insurance industry. The author shows that reputation campaigns may have a pro-cyclic effect, leading to more severe future reputation crises.

Abiola (2023) states that insurance losses, risks and premium calculation or pricing have been active and essential topics in insurance and actuarial literature, but most of this literature did not only stand the test of time due to the dynamic nature of insurance principles and practices in highly evolving environment but also lack the intuitive and detailed standard rating logic to adjust loss rating to a particular experience. There is a need to strike a balance in charging an appropriate and equitable premium by applying a suitable loss model that gives a sufficient uniquely determined solution that will not necessarily put an insurer or the insured in uncertain awkward business situations. Therefore, this research aims to obtain sufficient conditions for the convergence of the algorithm towards a fixed point under typical insurance loss and actuarial circumstances to achieve a uniquely determined solution. At the end, a unique fixed point was determined and the algorithm formulated converges towards that point through straightforward and simplified generalized formulae and functions.

Other insurance aspects like deposit insurance, premium rating of an endowment policy, control problem arising in insurance, Long Term Care, re-rating in general insurance, optimization problems of a household, and reinsurance game are dealt with in the following papers. These show how diverse problems fixed point theory can be used as a solution aid.

Kendall (1992) treats the fair deposit insurance premium as a fixed point of the value of insurance per dollar of deposits. Using the standard model of the value of deposit insurance and treating the premium as an up-front cost to a bank, it is shown that the fixed point premium exists and is unique under fairly general conditions. It is shown that ignoring the premium as an up-front cost may lead to underestimation of the fair premium. In addition, the fixed point model suggests that premium rates should vary with the ratio of deposits to total liabilities.

Bacinello (2003) analyzes a life insurance endowment policy, paid by annual premiums, in which the benefit is annually adjusted according to the performance of a special investment portfolio and a minimum return is guaranteed to the policyholder. In particular, the author considers both the case in which the annual premium is constant and the case in which the premium also is adjusted according to the performance of the reference portfolio. Moreover, the policy under scrutiny is characterized by the presence of a *surrender option*, that is, of an American-style put option that enables the policyholder to give up the contract and receive the *surrender value*. The paper aims to give sufficient conditions under which a (unique) fair premium exists. This premium is implicitly defined by an equation (or, alternatively, can be viewed as a fixed point of a suitable function) based on a recursive binomial tree by Cox et al. (1979). An iterative algorithm is then implemented to compute it.

Huang et al. (2012) point out that implicit methods for Hamilton-Jacobi-Bellman (HJB) partial differential equations give rise to highly nonlinear discretized algebraic equations. The classic policy iteration approach may not be efficient in many circumstances. In their article, Huang et al. derive sufficient conditions to ensure convergence of a combined fixed point policy iteration scheme for the solution of discretized equations. Numerical examples are included for a singular stochastic control problem arising in insurance (a guaranteed minimum withdrawal benefit), where the underlying risky asset follows a jump diffusion, and an American option assuming a regime switching process.

Long-term care (LTC) is mainly provided by the family and subsidiarily by the market and the government. To understand the role of these three institutions, it is important to understand the motives and the working of family solidarity. Canta and Pestieau (2013) focus on the case when children provide LTC to their dependent parents out of some norm that has been inculcated to them during their childhood by some exemplary behaviour of their parents towards their own parents. In the first part, they look at the interaction between the family and the market in providing for LTC. The key parameters are the probability of dependence, the probability of having a norm-abiding child and the loading factor. In the second part, they introduce the government which has a double mission: correct for a prevailing externality and redistribute resources across heterogeneous households. In their calculations of a stationary equilibrium of the intergenerational game they use Brouwer fixed point theorem.

In Borogovac (2016) a system of difference equations is introduced. It is developed for re-rating purposes in general insurance. A nonlinear transformation φ of a d-dimensional ($d \geq 2$) Euclidean space is introduced that enables us to express the system in the form $f^{t+1} := \varphi(f^{t})$, $t = 0, 1, 2,. \ldots$. Under typical actuarial assumptions, the existence of solutions of that system is proven using Brouwer's fixed point theorem in normed spaces. In addition, conditions that guarantee the uniqueness of a solution are given.

Liu et al. (2021a) study an optimisation problem of a household under a contagious financial market. The market consists of a risk-free asset, multiple risky assets and a life insurance product. The clustering effect of the market is modelled by mutual-excitation Hawkes processes where the jump intensity of one risky asset depends on both the jump path of itself and the jump paths of other risky assets in the market. The labor income is generated by a diffusion process. The goal of the household is to maximise the expected utilities from both the instantaneous consumption and the terminal wealth if he survives up to a fixed retirement date. Otherwise, a lump-sum heritage will be paid. The optimal strategies are obtained through the dynamic programming principle and by developing an iterative scheme to numerically solve the value function using Banach fixed point theorem.

Liu et al. (2021b) consider the optimal asset allocation, consumption, and life insurance strategies for a household with an exogenous stochastic income under a self-contagious market. Jump intensities of the risky asset depend on the history path of that asset. In addition to the financial risk, the household is also subject to an uncertain lifetime and a fixed retirement date. A lump-sum payment will be paid as a heritage if the wage earner dies before retirement. Under the dynamic programming principle, explicit solutions of the optimal controls are obtained when asset prices follow special jump distributions. We develop an iterative numerical scheme for more general cases to derive the optimal strategies. We also prove the existence and uniqueness of the solution to the fixed point equation and the convergence of an iterative numerical algorithm.

Liu et al. (2022) consider a stochastic asset allocation and reinsurance game between two insurance companies with contagious claims, where the insurance claim of one insurer can simultaneously affect the claim intensities of itself and its competitor. It is assumed that the insurance company's management wants to maximize the expected utility of the relative difference between its terminal surplus and that of its competitor at a fixed time point. The Nash equilibrium strategies are constructed by solving the Hamilton–Jacobi–Bellman equations, where the explicit formulas of the optimal allocation policies have been derived to be independent of the claim intensities. An iterative scheme is introduced based on the Feynman–Kac formula to compute the optimal proportional reinsurance policies numerically, where the existence and uniqueness of the solution to the fixed point equation and the convergence of the iterative numerical algorithm are proved rigorously.

BANKING AND FINANCE APPROACHES

An extensive textbook on fixed point theory and theoretical economics is the following: McLennan (2018) provides comprehensive coverage of fixed point theory at the highest possible level of generality. The book requires no mathematical knowledge beyond that common to all theoretical economists. Graduate students and researchers in economics, and related fields in mathematics and computer science, benefit from this book, both as a useful reference and as a well-written rigorous exposition of foundational mathematics. Numerous problems sketch key results from a wide variety of topics in theoretical economics, making the book an outstanding text for advanced graduate courses in economics and related disciplines.

Financial systems are discussed in the following two papers. Theory and methods outside fixed points are used. Chen (2008) examines the two most attractive characteristics, memory and chaos, in simulations of financial systems. A fractional-order financial system is proposed in this study. It is a generalization of a dynamic financial model earlier reported in the literature. The fractional-order financial system displays many interesting dynamic behaviors, such as fixed points, periodic motions, and chaotic motions. It has been found that chaos exists in fractional-order financial systems with orders less than 3. In this study, the lowest order at which this system yielded chaos was 2.35. Period doubling and intermittency routes to chaos in the fractional-order financial system were found.

Saha and Kavitha (2022) consider a large random network in which the performance of a node depends upon that of its neighbors and some external random influence factors. This results in random vector-valued fixed point (FP) equations in large dimensional spaces, and the authors study their almost-sure solutions. An underlying directed random graph defines the connections between various components of the FP equations. They obtain finite dimensional limit FP equations in a much smaller dimensional space, whose solutions aid in approximating the solution of FP equations for almost all realizations as the number of nodes increases. They use Maximum theorem for non-compact sets to prove this convergence, and apply the results to study systemic risk in an example financial network with many heterogeneous entities. The authors illustrated the accuracy of the approximation using exhaustive Monte-Carlo simulations. The approach can be utilized for a variety of financial networks (and others); the developed methodology provides approximate small-dimensional solutions to large-dimensional FP equations that represent the clearing vectors in the case of financial networks.

Dynamical systems are studied in the two following papers. Much theory outside fixed points is again used. Rinn et al. (2015) propose combining cluster analysis and stochastic process analysis to characterize high-dimensional complex dynamical systems by few dominating variables. As an example, stock market data are analyzed for which the dynamical stability as well as transitions between different stable states are found. This combined method allows especially to set up new criteria for merging clusters to uncover dynamically distinct states. The low-dimensional approach allows to recover the system's high-dimensional fixed points through an optimization procedure.

Syniavska et al. (2019) study the current issue of the counteracting cyberattacks in the banking sector, in particular in the field of e-banking. The main types of banking fraud, which are carried out in the online sphere, are considered. The authors propose a mathematical model that describes the process of counteracting e-banking fraud. The proposed model is based on the classic Lotka-Volterra model with logistic growth and the Holling-Tanner dynamic models. The fixed points of a dynamic system were calculated and analysed. It was determined that there are 4 possible types of fixed points. The constructed model could be used for theoretical study, and different simulation experiments with changing input parameters could be done. Unfortunately, it is difficult to investigate this question on real data, since the statistics on cyberattacks are closed.

Banking networks are discussed by the two following authors. Hurd and Gleeson (2011) introduce a probabilistic framework that represents stylized banking networks and aims to predict the size of contagion events. In contrast to previous work on random financial networks, which assumes independent connections between banks, the possibility of disassortative edge probabilities (an above average tendency for small banks to link to large banks) is explicitly incorporated. The authors give a probabilistic analysis of the default cascade triggered by shocking the network. They find that the cascade can be understood as an explicit iterated mapping on a set of edge probabilities that converges to a fixed point. A cascade condition is derived that characterizes whether or not an infinitesimal shock to the network can grow to a finite size cascade. It provides an easily computed measure of the systemic risk inherent in a given banking network topology. An analytic formula is given for the frequency of global cascades, derived from percolation theory on the random network. Two simple examples are used to demonstrate that edge-assortativity can have a strong effect on the level of systemic risk as measured by the cascade condition. Although the analytical methods are derived for infinite networks, large-scale Monte Carlo simulations demonstrate the results' applicability to finite-sized networks. Finally, they propose a simple graph theoretic quantity, which they call "graph-assortativity", that seems to best capture systemic risk.

Gauthier et al. (2012) points out that when setting banks' regulatory capital requirement based on their contribution to the overall risk of the banking system we have to consider that the risk of the banking system as well as each bank's risk contribution changes once bank equity capital gets reallocated. The authors define macroprudential capital requirements as the fixed point at which each bank's capital requirement equals its contribution to the risk of the system under the proposed capital requirements. They use a network based structural model to measure systemic risk and how it changes with bank capital and allocate risk to individual banks based on five risk allocation mechanisms used in the literature. Using a sample of Canadian banks, the authors find that macroprudential capital allocations can differ by as much as 25% from observed capital levels, are not trivially related to bank size or individual bank default probability, increase in interbank assets, and differ substantially from a simple risk attribution analysis. They further find that across all risk allocation mechanisms, macroprudential capital requirements reduce individual banks' default probabilities and the probability of a systemic crisis by about 25%. Macroprudential capital requirements are robust to model risk and are positively correlated to future capital raised by banks and future equity value losses. These results suggest that financial stability can be substantially enhanced by implementing a systemic perspective on bank regulation.

Other theoretical financial techniques are covered by the following four authors. Cheng and Robertson (2017) consider the problem of identifying current coupons for agency-backed to-be-announced pools of residential mortgages. Such coupons, or mortgage origination rates, ensure par-valued pools. In a doubly stochastic reduced form model which allows prepayment intensities to depend upon current and origination mortgage rates and underlying investment factors, they identify the current coupon as a solution to a degenerate elliptic, nonlinear fixed point problem. Using Schaefer fixed point theorem (see Voutilainen, 2022), Cheng and Robertson prove existence of a current coupon. They also provide an explicit approximation to the fixed point, valid for compact perturbations off a baseline factor-based intensity model. A numerical example shows that the approximation performs well in estimating the current coupon.

Kraft et al. (2017) study continuous-time optimal consumption and investment with Epstein-Zin recursive preferences in incomplete markets. They develop a novel approach that constructs the solution of the associated Hamilton-Jacobi-Bellman equation by a fixed point argument and makes it possible to compute both the indirect utility and, more importantly, optimal strategies. Based on these results, they also establish a fast and accurate method for numerical computations. This setting is not restricted to affine asset price dynamics; but only require boundedness of the underlying model coefficients.

Under non-exponential discounting, Yu-Jui and Nguyen-Huu (2018) develop a dynamic theory for stopping problems in continuous time. Their framework covers discount functions that induce decreasing impatience. Due to the inherent time inconsistency, they look for equilibrium stopping policies formulated as fixed points of an operator. Under appropriate conditions, fixed-point iterations converge to equilibrium-stopping policies. This iterative approach corresponds to the hierarchy of strategic reasoning in game theory and provides "agent-specific" results: it assigns one specific equilibrium-stopping policy to each agent according to her initial behaviour. In particular, it leads to a precise mathematical connection between naive and sophisticated behavior. This theory is illustrated in a real options model.

When hit with an adverse shock, banks that do not comply with capital regulation sell risky assets to satisfy their solvency constraint. When financial markets are imperfectly competitive, this creates a Generalized Nash Equilibrium Problem. Braouezec and Kiani (2023) consider a new framework with an arbitrary number of banks and assets, and show that Tarski's fixed point theorem can be used to prove the existence of a Nash equilibrium when markets are sufficiently competitive. They also prove the existence of ϵ-Nash equilibria.

CONCLUSION

In the field of fixed point theory, there have been both theoretical and practical innovations. For an example of the first type, Raj and Piramatchi (2020) introduced a notion of best proximity points in arbitrary topological spaces and established a few best proximity point theorems. They also extended the notion of contractive mapping to an arbitrary topological space and obtained fixed point theorems on topological spaces. For examples of the second type, see the chapters above and Voutilainen (2022).

Fixed point iterations are essential for solving fixed point problems, and they are presented in many cases above. They require the concept of convergence present in, e.g., Banach spaces, and most fixed point problems are formulated in Banach or Hilbert spaces or even Euclidean spaces. The new theory of Raj and Piramatchi (2020) study fixed points in general topological spaces where convergence is absent.

Along new theoretical innovations new applications emerge in various areas within pure mathematics, applied mathematics and e.g., economics.

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