Sortino(γ): A Modified Sortino Ratio With Adjusted Threshold

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A portfolio's Sortino ratio is strongly affected by the risk-free vs. risky assets mix, except for the case where the threshold, T is equal to the risk-free rate. Therefore, if T differs from the risk-free rate, the portfolio's Sortino ratio could potentially be increased by merely changing the mix of the risk-free and the risky components. The widely used Sharpe ratio, on the other hand, does not share this caveat.

We introduce a modified Sortino ratio, Sortino(γ), which is invariant concerning the portfolio's risk-free vs. risky assets mix and eliminates the above deficiency. The selected threshold T(γ), mimics the portfolio composition in the sense that it equals to the risk-free rate plus γ times the portfolio's equity risk premium. Higher selected γ reflects higher risk/loss aversion. We propose a procedure for optimizing the composition of the risky portion of the portfolio to maximize the Sortino(γ) ratio. In addition, we show that Sortino(γ) is consistent with first and second-order stochastic dominance with riskless asset rules.

Keywords: performance ratios, Sortino ratio, risk aversion, loss aversion, FSDR rule, SSDR rule

HIGHLIGHTS

- We introduce a modified Sortino ratio, Sortino(γ), whose threshold T(γ) is tied to the portfolio mix of risk-free vs. risky assets.
- Sortino(γ) is invariant with respect to the portfolio's of risk-free vs. risky assets mix. Therefore, it can be maximized only by improving the composition of the portfolio's risky component, and a maximization process is presented.
- Sortino(γ) is consistent with first and second stochastic dominance with riskless asset rules.

INTRODUCTION

The standard deviation (StDev) of returns is a proper measure of risk only in the limited case of normal return distributions. For all other distributions, preference by the mean variance criterion (MVC) that uses the StDev as its risk measure, is neither necessary nor sufficient condition for preference by all expected utility investors[1]. Indeed, the StDev as a measure of risk has been heavily criticized by many, including Markowitz (1959, pp. 286-288), the originator of the application of the MVC to portfolio optimization. Thus, many researchers suggested the replacement of the StDev with downside risk measures[2]. However, despite its deficiencies and the heavy criticism, the StDev is the risk measure employed by the Sharpe ratio, which is probably the most popular performance ratio, and it is also the risk factor in the well-known Capital Asset Pricing Model (CAPM)[3]. Its popularity is probably due, at least in part, to the simple mathematical algorithm needed to construct the optimal portfolios that minimize StDev for any given vector of expected returns given the variance-covariance matrix, as well as due to the resulting independence between of the portfolio's optimal risky assets composition and the degree to which the portfolio uses the riskless asset for lending and/or borrowing (i.e., the monetary Separation property).

One of the commonly used downside performance ratios, is the Sortino ratio. The numerator of the Sortino ratio is the expected return of the risky portfolio minus a defined threshold, *T*. The denominator is the root of the expected squared return deviations below *T*[4]. Unfortunately, where T differs from the risk-free rate, the Sortino ratio of a portfolio is affected by the risk-free vs. risky assets mix. This effect increases with the deviation of *T* from the riskless rate[5]. Thus, in the case where *T* differs from the risk-free rate, a portfolio's Sortino ratio is sensitive to its equity level. The optimal composition of the equity components of the portfolio cannot be separated from its optimal mix between the risky and the risk-free component. Our paper presents a modified Sortino ratio, Sortino(γ), invariant to the portfolio's equity level, for all relevant threshold values.

Our modification is based on replacing the constant *T* threshold, which is not responsive to the portfolio's equity level, with $T(\gamma)$ which equals the weighted average of the portfolio's expected rate of return and the risk-free rate, using weights of γ and (1-γ), respectively. Under the trivial assumption that the portfolio's expected rate of return exceeds the risk-free rate, the higher the γ the higher is $T(\gamma)$.

The paper is organized as follows. In Section 2 we show that if the conventional threshold *T* is below the portfolio's expected return but differs from the risk-free rate, the Sortino ratio increases or decreases monotonically and respectively with the portfolio's proportion of the risky vs. the riskless component. This undesired feature of a performance measure potentially allows portfolio managers to increase the ex-ante ratio by merely changing its equity level, namely, by altering the mix of the riskless vs. risky assets rather than by improving the composition of the portfolio's risky component. In Section 3 we present the modified performance measure, Sortino(γ), which employs the threshold $T(\gamma)$. In this section we show that the resulting ratio is invariant concerning the portfolio's split between the risky and riskless components[6]. Section 4 presents the procedure for obtaining the optimal risky portfolio which maximizes Sortino(γ) for a given γ. Section 5 shows that stochastic dominance with riskless asset rules (FSDR and SSDR) implies dominance by $S(\gamma)$. Dominance by SDR rules compares preferences for all expected utility investors with none-decreasing utility (FSDR) and for all investors with none-decreasing utility as well as none-increasing marginal utility (SSDR) provided they can borrow and lend against the risky portfolio using the same given riskless rate. Section 6 presents a summary and offers some conclusions.

SORTINO RATIO AND THE LEVEL OF THE EQUITY COMPONENT

The ex-ante Sortino ratio of a portfolio with a threshold T is given by[7]:

$$S_P(T) = \frac{E(\tilde{R}_P)-T}{\left[\underset{\tilde{R}_P\leq T}{E}(T-\tilde{R}_P)^2\right]^{0.5}} \tag{1}$$

$S_P(T)$ is the portfolio's Sortino ratio, $\tilde{R}_P$ is the (random) rate of return on the portfolio and E is the expected value operator. While the riskless rate of return is perhaps the most likely choice for a threshold, thresholds which are higher or lower than the riskless rate are used in the literature[8]. Denote the portfolio's proportion of the risky asset and the proportion of the risk-free asset by α and (1-α), respectively, and let $\tilde{R}_e$ and R_f represent the (random) rate of return on the equity component and the rate of return on the risk-free asset, respectively. Since $\tilde{R}_P = \alpha\tilde{R}_e + (1-\alpha)R_f$ we can rewrite Eq. (1) as follows:

$$S_P(T) = \frac{\alpha E(\tilde{R}_e) + (1-\alpha)R_f - T}{\left\{\underset{\alpha\tilde{R}_e + (1-\alpha)R_f \le T}{E}\left[T - \left(\alpha\tilde{R}_e + (1-\alpha)R_f\right)\right]^2\right\}^{0.5}} \quad (2)$$

Proposition 1 below, shows that the traditional Sortino ratio is invariant with respect to α when the threshold rate is equal to the risk-free rate.

Proposition 1. *If $T = R_f$, then for all α, $S_P(T) = S_e(T)$ which is the Sortino ratio of the all-equity portfolio (i.e, $\alpha = 1$):*

$$S_P(T = R_f) = S_e(T = R_f) = \frac{E(\tilde{R}_e) - R_f}{\left[\underset{\tilde{R}_e - R_f \le 0}{E}(R_f - \tilde{R}_e)^2\right]^{0.5}} \quad (3)$$

The proof of the proposition is immediate as Eq. (2) is reduced to Eq. (3) when $T = R_f$.

However, when $T \neq R_f$ and also $T < E(\tilde{R}_P)$ and $\alpha > 0$ then, Proposition 2 holds:

Proposition 2. *Given that $T < E(\tilde{R}_P)$ and $\alpha > 0$ then, $S_P(T) < S_P(R_f)$ and increases with α, if and only if, $T > R_f$. The opposite holds for $T < R_f$. The proof is presented in an Appendix.*

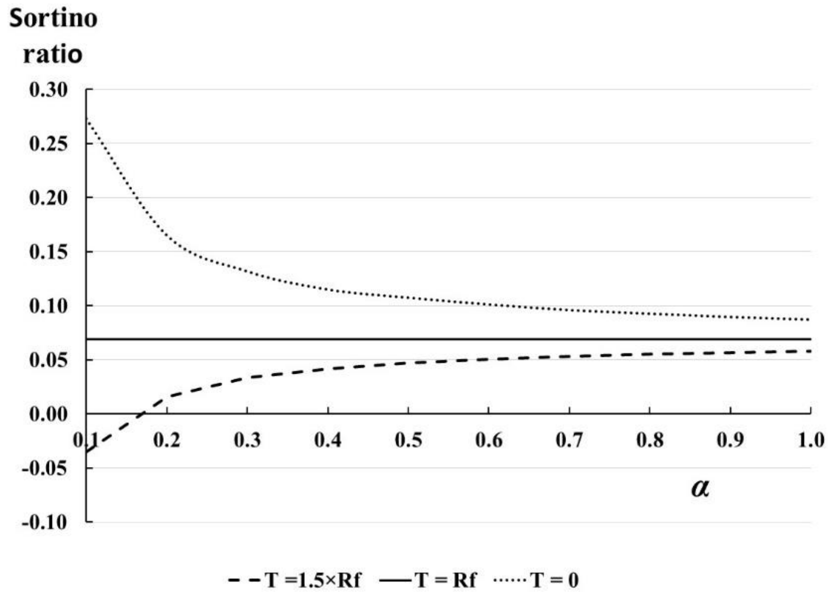
Note that the condition $T < E(\tilde{R}_P)$ guarantees a threshold below the expected return of the portfolio and the condition $\alpha > 0$ eliminates an overall short position of the portfolio. These are two very reasonable requirements.

Figure 1 presents estimated Sortino ratios using bootstrapping simulations on the S&P-500 index rates as a function of α. The data and simulations details are in the Figure's caption.

It is clear from Proposition 2 and Figure 1 that the selection of T below (above) the risk-free rate, may lead fund managers who seek to increase their fund's Sortino ratio, to adopt too low (high) equity investment strategy. The Sortino ratio is particularly sensitive to changes of α at low α levels.

In the next section, we present our modified Sortino(γ) ratio which employs a threshold $T(\gamma)$ that equals γ times the expected return of the portfolio and (1-γ) times the risk-free rate. It is shown that the modified Sortino ratio is invariable concerning the proportion of risk-free asset in the portfolio.

FIGURE 1
SORTINO RATIO AS A FUNCTION OF *α* WITH THREE ALTERNATIVE THRESHOLD VALUES



Based on 2000 random draws from120 monthly returns on the S&P-500 index, February 2008 to January 2018.

The economic logic for choosing $T(\gamma)$ as a threshold rate, is that the threshold for measuring the downside risk of a portfolio should be adjusted to the portfolio's risk premium. This is because it is likely that as the selected overall expected volatility of the portfolio increases, the investor's propensity to absorb losses increases as well. We thus define the threshold rate in terms of the risk-free rate plus γ times the portfolio's expected premium above the risk-free rate. If $\gamma = 1$, any return lower than the expected portfolio return is considered in the "loss" region when calculating the downside risk. If, for example, $\gamma = 0.5$, the downside risk measure considers all the returns which are lower than the risk-free rate plus 50% of the portfolio's expected risk premium, and when $\gamma = 0$, all returns below the risk-free rate are regarded as a loss and count as part of the downside risk measure. Negative γ values maybe unlikely for rational investors because they place the threshold below the riskless rate while the riskless rate is always an open alternative and ignoring the loss between the risk-free rate and the threshold, even when the latter is greater than 0, may be supported, at most, on psychological grounds. However, if a negative γ is selected, such as $\gamma = -0.2$, the downside risk measure considers only the returns which are lower than the risk-free rate minus 20% of the portfolio's risk premium. Non-positive thresholds exist for the following γ values:

$$\gamma \leq -\frac{R_f}{E(\tilde{R}_P)-R_f} \tag{4}$$

Proposition 3 prove that Sortino(γ) is invariant with respect to α.

Proposition 3. *S(γ) ratio is invariant with respect to the portfolio's equity level, α.*
Proof:

$$S(T(\gamma)) \equiv S(\gamma) = \frac{E(\tilde{R}_P)-T(\gamma)}{\left\{\underset{\tilde{R}_P<T(\gamma)}{E}[T(\gamma)-\tilde{R}_P]^2\right\}^{0.5}} \tag{5}$$

Eq. (5) can be specified as:

$$S(\gamma)=\frac{\alpha\left[\left(E(\tilde{R}_e)-R_f\right)-\gamma\left(E(\tilde{R}_e)-R_f\right)\right]}{\left\{\underset{\alpha(\tilde{R}_e)+(1-\alpha)R_f<\gamma\alpha(E(\tilde{R}_e)-R_f)+R_f}{E}\left[\left(\gamma\alpha(E(\tilde{R}_e)-R_f)+R_f\right)-\left(\alpha(\tilde{R}_e)+(1-\alpha)R_f\right)\right]^2\right\}^{0.5}} \tag{6}$$

which is the same as:

$$S(\gamma)=\frac{(1-\gamma)\left[E(\tilde{R}_e)-R_f\right]}{\left\{\underset{\tilde{R}_e-R_f<\gamma(E(\tilde{R}_e)-R_f)}{E}\left[\gamma\left(E(\tilde{R}_e)-R_f\right)-\left(\tilde{R}_e-R_f\right)\right]^2\right\}^{0.5}} \tag{7}$$

The last formulation of $S(\gamma)$ is invariant concerning α as claimed by the proposition. If γ is positive (zero) the threshold is set higher than (equal to) the risk-free rate. Negative γ implies threshold below the risk-free rate.

THE PORTFOLIO'S OPTIMAL RISKY COMPONENT FOR A GIVEN γ

Since S(γ) is invariant concerning α, its ex-ante maximization can be attained only by changing the composition of the portfolio's risky component. Define the equity "risk premium ratio" as the ratio of the (random) equity component's risk premium to its expected value, and denote it $\widetilde{rpr_e}$:

$$\widetilde{rpr_e}=\frac{\tilde{R}_e-R_f}{E(\tilde{R}_e)-R_f} \tag{8}$$

Proposition 4. *For a given γ the ratio S(γ) is maximized by minimizing the expected downside square deviations of the "risk premium ratio" from γ, namely:*

$$\underset{\underline{q}}{MIN}\left[\underset{\tilde{R}_e<\gamma E(\tilde{R}_e)+(1-\gamma)R_f}{E}\left(\widetilde{rpr_e}-\gamma\right)^2\right] \tag{9}$$

where $\underline{q}$ is the vector of the proportions invested in the individual risky securities.

The proof is based on Eq. (9) that can be re-written as:

$$S(\gamma)=\frac{1-\gamma}{\left[\underset{\tilde{R}_e<\gamma E(\tilde{R}_e)+(1-\gamma)R_f}{E}\left(\gamma-\widetilde{rpr_e}\right)^2\right]^{0.5}} \tag{10}$$

As argued, the conventional Sortino ratio is not invariant with respect to α (except for $T = R_f$); therefore, its reward vs. downside risk frontier also changes with α. In contrast, $S(\gamma)$ is invariant to the choice of α and therefore one may apply Eq. (11) subject to any given expected return and obtain the minimum downside risk for each expected return, thereby creating the efficient mean-downside risky frontier of the risky portion of the portfolio for the chosen γ. Consequently, for any $T(\gamma)$, one can use the minimization process in Eq. (11) to find the optimal composition of the risky component of the portfolio. The portfolio's optimal split between the risk-free asset and the optimal risky component is determined subjectively by the investor. Figure 2 depicts the result of this optimization process: it represents tradeoffs for a given positive γ. The portfolio's optimal risky component, composed only with the equities, has an expected return of $E(\tilde{R}_e^*)$. This optimal portfolio is determined objectively and applies only to investors who select a specific γ. The overall optimal subjective combination of the risky assets and the risk-free asset for the investor who selected the said γ, has an expected rate of return $E(\tilde{R}_P^*)$. The optimal overall portfolio is found at the tangency point between the investor's relevant indifference curve and the tangent line that run from R_f toward (and beyond) the tangency point with the efficient risky frontier at point O.

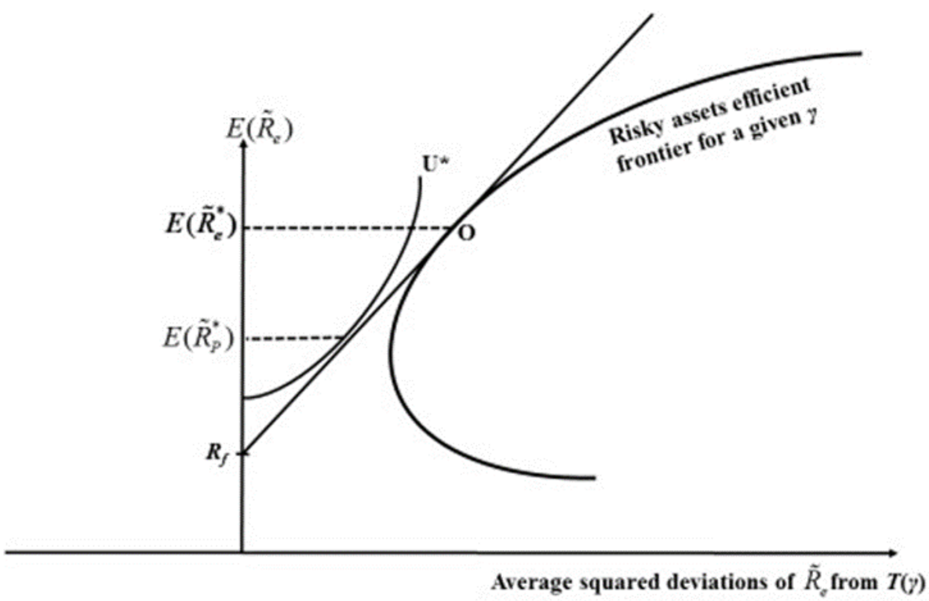
CONSISTENCY WITH STOCHASTIC DOMINANCE WITH RISKLESS ASSET RULES (SDR)

Stochastic dominance (SD) rules provide necessary and sufficient conditions for preference between any two alternative return (or income) distributions, $\tilde{X}$ and $\tilde{Y}$, for a wide range of assumptions regarding the investor's utility function. The First-degree Stochastic Dominance (FSD) rule assumes only a non-decreasing utility function, while the Second-degree Stochastic Dominance (SSD) rule also assumes a non-increasing marginal utility, i.e., risk aversion. The SD rules are partial ordering rules since, in general, it is not guaranteed that all investors with the assumed utilities prefer the same one alternative over another.

Let the preference of $\tilde{X}$ over $\tilde{Y}$ by the conventional Sortino ratio *S*(*T*), be denoted as $\tilde{X} \underset{S(T)}{\geq} \tilde{Y}$ and let the same preference by S(γ), be denoted $\tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$. These preferences present complete ordering which, a-priori, may be inconsistent with SD rules. Namely, in general, Sortino ordering may not be sufficient for dominance by SD rules. With respect to *S*(*T*), Balder and Schweizer (2017) (BS) showed that if $\tilde{X} \underset{SSD}{D} \tilde{Y}$ and $E(\tilde{X}) \geq T \geq E(\tilde{Y})$ then $\tilde{Y} \underset{S(T)}{\geq} \tilde{X}$.

Levy and Kroll (1976) extended the SD rules to portfolios of risky assets that could be diversified with the riskless asset. They denoted these rules SDR rules (i.e., Stochastic Dominance with Riskless asset rules). The First and Second degree SDR rules, are denoted FSDR and SSDR rules, respectively. They proved that if there is a combination of a proportion α invested in $\tilde{X}$ and (1-α) invested in the riskless asset such that this combination dominates $\tilde{Y}$ by FSD or SSD, then for any other combination of $\tilde{Y}$ with the riskless asset, there is at least one other combination of $\tilde{X}$ with the riskless asset that dominates it by FSD or SSD, respectively.

FIGURE 2
THE EFFICIENT RISKY FRONTIER, THE OPTIMAL EXPECTED RATE OF RETURN OF THE PORTFOLIO'S RISKY COMPONENT, $E(\tilde{R}_e^*)$, AND THE OPTIMAL EXPECTED RATE OF RETURN OF THE OVERALL PORTFOLIO $E(\tilde{R}_p^*)$, FOR A CHOSEN γ



It should be noted that the partial ordering by SDR rules is potentially much more effective than the SD rules. For example, assume that $\tilde{X}$ and $\tilde{Y}$ are uniformly distributed returns: $\tilde{X} \sim U(0,20)$ and $\tilde{Y} \sim U(5,10)$. In this example, there is no FSD or SSD dominance relationships between $\tilde{X}$ and $\tilde{Y}$. The expected return of

$\tilde{X}$ is greater than that of $\tilde{Y}$ (10 > 7.5) hence $\tilde{Y}$ clearly does not dominate $\tilde{X}$, but also the lowest outcome of $\tilde{X}$ is smaller than that of $\tilde{Y}$ (0 < 5) and thus $\tilde{X}$ does not dominate $\tilde{Y}$. However, if each of the risky assets could be diversified with a risk-free asset whose return is 7.2%, then, for example, a portfolio of 30% $\tilde{X}$ and 70% R_f is also distributed uniformly, $\tilde{X}_{\alpha=30\%} \sim U(5.04{,}11.04)$, and it dominates $\tilde{Y}$ by FSD. Likewise, by SDR rules, for any combination of $\tilde{Y}$ and R_f, one can find at least one combination of $\tilde{X}$ with R_f that dominates it.

This example shows that considering diversification between risky and risk-free alternatives, a lack of dominance by the FSD or SSD rules between two distributions may nevertheless exhibit dominance relationship by the FSDR or SSDR rules, respectively.

Proposition 5. *If* $\tilde{X} \underset{FSD}{D} \tilde{Y}$ *and there are no short sales of either* $\tilde{X}$ *or* $\tilde{Y}$*, then* $\tilde{X} \underset{S(T)}{\geq} \tilde{Y}$ *for every T and by* $\tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$ *for every γ.*

Proof: The proof is almost immediate. Such dominance implies that for each cumulative distribution of order P $(0 \leq P \leq 1)$ the $\tilde{X}(P) \geq \tilde{Y}(P)$. Thus, for each constant T or $T(\gamma)$ as calculated by Eq. (4) we have $T - \tilde{X}(p) \leq T - \tilde{Y}(p)$. Denote by $P_{\tilde{X}}(T)$ and $P_{\tilde{Y}}(T)$) the P order probabilities that lead to the T value of $\tilde{X}$ and $\tilde{Y}$, respectively. Due to the FSD assumption, also $P_{\tilde{X}}(T) \leq P_{\tilde{Y}}(T)$ and thus the average square deviations between T and $\tilde{X}$, is also smaller than the respective average square deviations between T and $\tilde{Y}$. Namely:

$$\int_0^{P_{\tilde{X}}(T)} p\big(T - \tilde{X}(p)\big)^2 dp \leq \int_0^{P_{\tilde{Y}}(T)} p\big(T - \tilde{Y}(p)\big)^2 dp \qquad (11)$$

Proposition 6. $\tilde{X} \underset{FSDR}{D} \tilde{Y} \Rightarrow \tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$ *for all γ < 1.*

Proof. If there is FSDR of $\tilde{X}$ over $\tilde{Y}$ then there is a combination of $\tilde{X}$ and the risk-free asset, that dominates a given combination of $\tilde{Y}$ with the risk-free asset, and thus we are back in a situation which is presented in Proposition 5. It is guaranteed that for any other combination of $\tilde{Y}$ with the risk-free asset there is at least one other combination of $\tilde{X}$ with the risk-free asset that dominates it, and the conditions of Proposition 5 hold again.

Proposition 7. $\tilde{X} \underset{SSDR}{D} \tilde{Y} \Rightarrow \tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$ *for all γ < 1.*

Proof. If there is SSDR of $\tilde{X}$ over $\tilde{Y}$ then there is a combination of $\tilde{X}$ and the risk-free asset, that dominates a given combination of $\tilde{Y}$ with the risk-free asset, and thus we are back in a situation which is presented in Proposition 5. It is guaranteed that for any other combination of $\tilde{Y}$ with the risk-free asset there is at least one other combination of $\tilde{X}$ with the risk-free asset that dominates it, and the conditions of Proposition 5 hold again.

CONCLUDING REMARKS

Sortino ratio is defined as the excess expected return over a given threshold T divided by the square root of the expected squared return deviations below T and it is one of the most popular downside performance measures among practitioners. Since investors vary concerning their attitude toward loss, they use different thresholds to define the rate that separates the loss from the reward and indeed the Sortino literature allows a wide range of T values. Our paper shows that if T is above (below) the riskless rate, the Sortino ratio increases (decreases) with a portfolio's equity level. This undesirable shortcoming allows one to increase the portfolio's degree of leverage.

Our modified Sortino ratio, uses the target $T(\gamma)$, which equals γ times the expected return of the portfolio plus (1- γ) times the risk-free rate. Since the expected ex-ante return of a risky portfolio is higher than the risk-free return, the threshold $T(\gamma)$, which reflects the investor's sensitivity to loss, increases with γ. In contrast to the conventional Sortino ratio, our modified ratio is invariant with respect to the portfolio's equity level, α, and depends only on the selected "loss benchmark" γ. Hence, an ex-ante change of Sortino(γ) ratio, for a given γ, is possible only through better composition of the risky portion of the portfolio.

The paper presents a simple criterion for minimizing the downside risk for any chosen expected return and γ, allowing the investor to separate the optimal mix of the risky and riskless components of the portfolio from the optimal composition of the portfolio's risky component.

We also show that ranking portfolios' performance by first and second degree stochastic dominance with riskless asset rules (FSDR and SSDR respectively), implies ranking by $S(\gamma)$. Stochastic dominance with riskless asset rules (SDR) examine dominance between risky portfolios where it is assumed that each of the distributions being compared may be diversified with the risk-free asset. These SDR rules potentially show dominance where stochastic dominance without riskless asset rules signal no dominance. Dominance by SDR rules implies dominance by our $S(\gamma)$ criterion.

ENDNOTES

1. The stochastic dominance rules for all rational investors (First degree Stochastic Dominance rule - FSD) and for all rational risk averse investors (Second degree Stochastic Dominance rule - SSD) provide necessary and sufficient (optimal) efficiency rules for preference. However, the practical application of these rules for constructing optimal portfolios and obtaining market equilibrium conditions is quite limited.
2. For a review of many downside risk measures, see Sortino and Price 1994, Sortino and Forsey 1996, Nawrocki 1999, Pedersen and Satchell 2002, Pedersen and Rudholm-Alfvin 2003, Sortino 2009.
3. The basic CAPM was developed by Treynor 1961, 1965, Sharpe 1964, Lintner 1965, and Mossin 1966).
4. The ratio belongs to a wider set of performance ratios, Kappa, that employ the lower partial moment as a measure of risk (Kaplan and Knowles 2004).
5. In what follows, and for the purpose of abbreviation, we often refer to the proportion of the portfolio's risky (equity) component as the "equity level" or the "risk level" of the portfolio.
6. In our theoretical model we assume that there is a riskless rate and it is the same for borrowing and lending and the same for the investment fund and the individual investors. Under these assumptions, a reasonable performance measure should not be affected by the selected proportion of the riskless asset vs. the risky assets in the portfolio, regardless of whether the choice is made by the fund manager or by the ultimate ("individual") investor.
7. The presentation below is an ex-ante version while in practice, the ratio is estimated using sample observations.
8. For example: Frugier (2016) and Hu et al. (2015) and others use 0% as a threshold. Booth and Broussard (2017) consider thresholds from -0.01 to -0.10, and Perelló (2007) examines thresholds from -30% to +30%.

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APPENDIX: PROOF OF PROPOSITION 2

We begin with the Sortino ratio of a two-asset portfolio consisting of a proportion α invested in a risky asset (equity) and a proportion (1-α) invested in a riskless asset:

$$S_P(T) \equiv S = \frac{E(\tilde{R}_P) - T}{\left\{\underset{\tilde{R}_P \le T}{E}(T - \tilde{R}_P)^2\right\}^{0.5}} =$$

$$= \frac{E\left[\alpha\tilde{R}_e + (1-\alpha)R_f\right] - T}{\left\{\underset{\alpha\tilde{R}_e + (1-\alpha)R_f \le T}{E}\left[T - \left(\alpha\tilde{R}_e + (1-\alpha)R_f\right)\right]^2\right\}^{0.5}}$$

Define $T \equiv R_F + \Delta$ and for $\alpha > 0$ we may write:

$$S = \frac{E\left(\tilde{R}_e - R_f - \frac{\Delta}{\alpha}\right)}{\left[\underset{\tilde{R}_e - R_f - \frac{\Delta}{\alpha} \le 0}{E}\left(\frac{\Delta}{\alpha} - \tilde{R}_e + R_f\right)^2\right]^{0.5}}$$

Denoting $u_\alpha = \tilde{R}_e - R_f - \frac{\Delta}{\alpha}$, we rewrite the ratio as:

$$S = \frac{E(\tilde{u}_\alpha)}{\left[\underset{\tilde{u}_\alpha \le 0}{E}(-\tilde{u}_\alpha)^2\right]^{0.5}}$$

$$E(\tilde{R}_P) > T \quad \Rightarrow \alpha E(\tilde{R}_e) + (1-\alpha)R_f > R_f + \Delta \quad \Rightarrow \alpha\left(E(\tilde{R}_e) - R_f\right) - \Delta > 0$$

$$\Rightarrow E(\tilde{R}_e) - R_f - \frac{\Delta}{\alpha} > 0 \Rightarrow E(\tilde{u}_\alpha) > 0$$

In addition, we note that $\frac{\partial E(u_\alpha)}{\partial \alpha} = \frac{\partial u_\alpha}{\partial \alpha} = \frac{\Delta}{\alpha^2}$.

For $\Delta \neq 0$ we obtain:

$$\frac{\partial S}{\partial \alpha} = \frac{\frac{\Delta}{\alpha^2}\left\{\left[\underset{\tilde{u}_\alpha \leq 0}{E}(-\tilde{u}_\alpha)^2\right]^{0.5} + E(\tilde{u}_\alpha) \times 0.5 \times \left[\underset{\tilde{u}_\alpha \leq 0}{E}(-\tilde{u}_\alpha)^2\right]^{-0.5} \times 2 \times \underset{\tilde{u}_\alpha \leq 0}{E}(-\tilde{u}_\alpha)\right\}}{\underset{\tilde{u}_\alpha \leq 0}{E}(-\tilde{u}_\alpha)^2}$$

The denominator of the derivative is clearly positive. The first term in the numerator's curly brackets is positive as well. The expected value of $\tilde{u}_\alpha$ is likewise positive as noted above. And the last term in the numerator of the curly brackets is positive by definition, which ensures that the entire expression inside the numerator's curly brackets is positive. It follows that the sign of the derivative, $\frac{\partial S}{\partial \alpha}$, is determined by the sign of Δ. Positive Δ indicates a threshold higher than the risk-free rate in which case the derivative is positive while negative Δ indicates a threshold lower than the risk-free rate in which case the derivative is negative.