Questions on Cost-Benefit Analysis and a Discussion of Present Answers: How Should One Determine the Social Discount Rate to Be Announced?

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In the wake of the CBA used for public investment programming in France, new recommendations have been formulated for projects whose advantages (costs or benefits) are exposed to macroeconomic risks (correlation with GDP per capita), distinguishing between classically Gaussian hazards and "rare disasters" à la Barro. Taking rare disasters into account significantly modifies the value of the discount rate, along with the mathematical expectation of the advantages, and accentuates the distinction between procyclical and counter-cyclical projects.

Keywords: macroeconomic risks on projects, procyclical versus counter-cyclical projects, which discount rate, which expected advantages

INTRODUCTION

In France, CBA has been used for a long time, and more particularly since WW2, to evaluate public investments, mainly in the field of energy and transportation, with progressive extension to other areas, like health for instance. A social discount rate has been set, for successive periods. In 2011, a commission chaired by Christian Gollier recommended to take account of macroeconomic risks in the appraisal of public investments and in 2013, a commission chaired by Emile Quinet specified how to implement these guidelines. In 2017, a Committee of experts, chaired by Roger Guesnerie, was appointed to update the methods of socioeconomic evaluation of public investments and in 2021, this Committee has recommended a revision of the social discount rate. The following contribution presents the proceedings of this revision.

SUMMARY

We follow the Ramsey approach, which is generally adopted in different countries. The model that we use is formulated in discrete time t, with $t = 0$ at a chosen reference date, for instance 2020. It rests on the announcement of the future macroeconomic growth stochastic forecast. It refers to W, which represents the expectation of an intertemporal monetized social welfare function, chosen as an additive function of the per capita GDP, added from $t = 0$ to infinity. It depends on two parameters: δ (pure rate of time preference) and γ (which determines the constant intra-period risk aversion). All monetized flows are expressed in euros of the reference year.

We then consider a small project at the margin of the GDP. Its advantage (benefits minus cost) is denoted A_t. A_t is a random variable, which is supposed to be connected to the random per capita GDP denoted Y_t, through the function $A_t = \underline{A}_t.Y_t^{\beta_t}$, where β_t is the elasticity coefficient and $\underline{A}_t$ a scale factor specific of the project. The advantage A_t is procyclical if $\beta_t > 0$, contracyclical if $\beta_t < 0$.

The socioeconomic net present value $SE\ NPV$ of the project is defined as the variation ΔW caused to W by this project. The component of SE NPV for the year t, denoted $SE\ NPV_t$, can also be written as the expected value of the advantage EA_t multiplied by the discount factor R_t, itself linked to the discount rate ρ_t by the formula $R_t = e^{-\rho_t.t}$.

The calculations of EA_t and ρ_t depend on the parameters β_t and $\underline{A}_t$, but too on the probability distribution of the random variable Y_t, or more precisely of the random variables $z_t = ln\ ln\ \frac{Y_t}{Y_{t-1}}$. In order to simplify, we assume that theses variables z_t are independent and identically distributed (*iid*), according to a type z (the per capita GDP evolves like in a random walk). We still have to specify the probability distribution of this random variable z.

A standard assumption is that z follows a gaussian distribution, the mean of which is denoted μ or k_1 (cumulant number 1) and the variance of which is denoted σ^2 or k_2 (cumulant number 2). Then, it follows that:

$EA_t = \underline{A}_t.e^{\nu_t.t}$ where $\nu_t = \beta_t.k_1 + \frac{\beta_t^2}{2}.k_2$ (polynomial of degree 2 in β_t)

$\rho_t = rf_t + \beta_t.\phi$ where $rf = \delta + \gamma.k_1 - \frac{\gamma^2}{2}.k_2$ and $\phi = \gamma.k_2$ (polynomial of degree 1 in β_t)

But this gaussian assumption appears unsatisfactory, on the one hand, with respect to the evolution of per capita GDP registered in France on the long past period and, on the other hand, with respect to the risks of the future. The Committee has deemed it more appropriate to take account of risks of rare disasters, in particular in the way suggested by Barro (2011).

More precisely, the variable z is supposed to be the sum of two random variables, independent: $z = za + zb$, where za is supposed gaussian and zb encompasses rare disasters à la Barro.

The calculations have been calibrated to stick to the French case: growth prospects to 2070 according to COR[1] 2020 projections; variance observed in France over the retrospective period 1913-2020; risk aversion increased from 2 to 2.5.

The analytical solutions for EA_t and ρ_t appear then to have forms less simple than the two polynomials above-mentioned. A solution would be to provide tables. Nevertheless, for the sake of simplicity, the Committee has deemed it preferable to provide formulas under similar simple forms, the coefficients of which being estimated to minimise the sum of quadratic error between the approximated outcomes and the analytical outcomes, using the least square method.

The model is presented in part 1, the calculations and outcomes in part 2. To summarize:

"With rare disasters", the formulas obtained according to the polynomial approximation are:

$\nu = \beta * 1.15 + \frac{\beta^2}{2}$*0.93 % p.a.
$\rho = 1.20 + \beta * 2.00\%$ % p.a.

"Without rare disasters (pure gaussian scenario)", the corresponding (exact) formulas are:

$\nu = \beta * 1.15 + \frac{\beta^2}{2}$*0.48 % p.a.
$\rho = 1.83 + \beta * 1.18$ % p.a.

Taking account of rare disasters has thus a significant impact on the rates ν and ρ. And one can check that the impact on the socioeconomic net present value of the project is all the more noticeable the lower β is below γ.

Other relevant concerns are listed below:

- Is the reference random growth path fixed, or subject to policy. uncertainty, reflecting the CBA it generates….
- How to take into account the increased uncertainty of the long run forecasts….
- How to improve the choice of the welfare parameters of the social welfare function…

Part 1: Presentation of the Model

1. According to Guesnerie G. and Ch. Gollier (2017) "Discussion sur l'actualisation : un arrière plan analytique simplifié", let us consider a Ramsey model with an intertemporal monetized social welfare function W, written in discrete time, with $t = 0$ at a chosen reference date, for instance 2020, and specified as follows:

$$W = \left[\sum_{t=0}^{t=+\infty} e^{-\delta.t}.EU(Y_t)\right]/U'(Y_0) \tag{1}$$

where: δ is the pure rate of time preference; E is the "mathematical expectation" operator; $U(.)$ measures social welfare; Y_t is the per capita GDP year t; $U'(Y_0)$ is the marginal utility of year 0. All monetized flows are expressed in euros of the reference year.
Y_t is regarded as a random variable, the probability distribution of which will be specified further.

2. Let us then consider a small project at the margin of the GDP trajectory. Its advantage (benefits minus costs) of the year t is denoted A_t. With project, the welfare function becomes:

$$W_{with} = \left[\sum_{t=0}^{t=+\infty} e^{-\delta.t}.EU(Y_t + A_t)\right]/U'(Y_0) \tag{2}$$

3. The socioeconomic net present value $SE\ NPV$ of the project is defined as the variation of W induced by this project:

$$SE\ NPV = W_{with} - W = \left[\sum_{t=0}^{t=+\infty} e^{-\delta.t}.\{EU(Y_t + A_t) - EU(Y_t)\}\right]/U'(Y_0) \tag{3}$$

Let us consider the first order approximation:

$$SE\ NPV = \left[\sum_{t=0}^{t=+\infty} e^{-\delta.t}.E[A_t.U'(Y_t)]\right]/U'(Y_0) \tag{4}$$

which can be written:

$$SE\ NPV = \sum_{t=0}^{t=+\infty} e^{-\delta.t}.E\left[A_t.\frac{U'(Y_t)}{U'(Y_0}\right] \tag{5}$$

4. Let us consider the component of SE NPV for the year t, denoted $SE\ NPV_t$:

$$SE\ NPV_t = \boldsymbol{e^{-\delta.t}.E\left[A_t.\frac{U'(Y_t)}{U'(Y_0}\right]} \tag{6}$$

This component can also be written as the expected value EA_t of the advantage of the year t, multiplied by the discount factor R_t, itself linked to the discount rate ρ_t by the formula $R_t = e^{-\rho_t.t}$:

$$SE\ NPV_t = e^{-\rho_t.t}.EA_t \tag{7}$$

The formula for the discount rate is therefore:

$$\rho_t = \delta - \frac{1}{t}.ln\, ln\, E \left[A_t.\frac{U'(Y_t)}{U'(Y_0}\right] + \frac{1}{t}.ln\, ln\, EA_t \tag{8}$$

5. Let us now suppose the function $U(Y_t)$ of the form CRRA[2]:

$$U(Y_t) = \frac{Y_t^{1-\gamma}-1}{1-\gamma} \tag{9}$$

γ is the risk aversion[3].

(9) implies: $U'(Y_t) = Y_t^{-\gamma}$ and in particular:

$$U'(Y_0) = Y_0^{-\gamma} \tag{10}$$

6. In addition, let us suppose that the advantage A_t is a random variable, which is supposed to be influenced by the random per capita GDP Y_t, through the function:

$$A_t = \underline{A}_t.Y_t^{\beta_t} \tag{11}$$

where β_t is the elasticity coefficient and $\underline{A}_t$ a scale factor specific of the project[4].

The advantage A_t is procyclical if $\beta_t > 0$, contracyclical if $\beta_t < 0$. The expectation is:

$$EA_t = \underline{A}_t.EY_t^{\beta_t} \tag{12}$$

7. According to (10) and (11):

(6) becomes:

$$SE\ NPV_t = \underline{A}_t.e^{-\delta.t}.E\left[\left(\frac{Y_t}{Y_0}\right)^{\beta_t-\gamma}\right] \tag{13}$$

(8) becomes:

$$\rho_t = \delta - \frac{1}{t}.ln\, ln\, E\ \left(\frac{Y_t}{Y_0}\right)^{\beta_t-\gamma} +\ E\ \left(\frac{Y_t}{Y_0}\right)^{\beta_t} \tag{14}$$

Let us consider the annual evolution z_t of the per capita GDP Y_t:

$$z_t = ln\frac{Y_t}{Y_{t-1}} \tag{15}$$

In other words: $z_t = ln\, ln\, (1+g_t)$, where g_t is the annual growth rate of Y_t.

Hence:

$$\frac{Y_t}{Y_0} = e^{\sum_{u=1}^{u=t} z_u} \tag{16}$$

In order to simplify, we assume that the random variables z_t are independent and identically distributed (*iid*), according to a type z (the per capita GDP evolves like in a random walk). Hence:

(12) becomes:

$$EA_t = \underline{A}_t . \left(Ee^{\beta_t . z}\right)^t \tag{17}$$

(13) becomes:

$$SE\ NPV_t = \underline{A}_t . e^{-\delta . t} . \left(Ee^{(\beta_t - \gamma) . z}\right)^t \tag{18}$$

Let us denote[5] the two following rates:

$$\nu_t = ln\ ln\ Ee^{\beta_t . z} \text{ and } \tau_t = -\ ln\ ln\ Ee^{(\beta_t - \gamma) . z} \tag{19}$$

Hence:

(17) is written:

$$EA_t = \underline{A}_t . e^{\nu_t . t} \tag{20}$$

(18) is written:

$$SE\ NPV_t = \underline{A}_t . e^{-(\delta + \tau_t) . t} \tag{21}$$

(14) is written:

$$\rho_t = \delta + \tau_t + \nu_t \tag{22}$$

8. We still have to specify the probability distribution of this random variable z. An elementary assumption would be that z follows a gaussian distribution. But this gaussian assumption appears unsatisfactory, on the one hand, with respect to the evolution of the per capita GDP registered in France on the long past period, and on the other hand, with respect to the risks of the future. The Committee has deemed it more appropriate to take into account risks of rare disasters, in particular as suggested by Barro (2011).

More precisely, the variable z is supposed to be the sum of two independent random variables:

$$z = za + zb \tag{23.1}$$

za is supposed gaussian and zb non-gaussian, encompassing rare disasters à la Barro (2011).
Thus:

$$\frac{Y_t}{Y_{t-1}} = e^{z_a} . e^{zb}, \tag{23.2}$$

a product of two factors, respectively e^{z_a} and e^{z_b}.
As za and zb are independent, it is clear that

$$Ee^{\beta_t . z} = Ee^{\beta_t . (za + zb)} = Ee^{\beta_t . za} . Ee^{\beta_t . zb} \tag{24.1}$$

Hence

$$ln\, ln\, Ee^{\beta_t . z} = ln\, ln\, Ee^{\beta_t . za} + ln\, ln\, Ee^{\beta_t . zb} \quad (24.2)$$

Finally, according to (19):

$$\nu_t = \nu a_t + \nu b_t \quad (25.1)$$

In the same way:

$$\tau_t = \tau a_t + \tau b_t \quad (25.2)$$

The discount rate is still given by:

$$\rho_t = \delta + \tau_t + \nu_t \quad (25.3)$$

9. Let us here consider the gaussian component za and denote its mean μa or ka_1 (cumulant number 1) and its variance σa^2 or ka_2 (cumulant number 2).

A classical outcome is that:

$$Ee^{\beta_t . za} = e^{\beta_t . ka_1 + \frac{\beta_t^2}{2} . ka_2}, \text{ thus: } \nu a_t = \beta_t . ka_1 + \frac{\beta_t^2}{2} . ka_2 \quad (26)$$

In the same way: $Ee^{(\beta_t - \gamma) . za} = e^{(\beta_t - \gamma) . ka_1 + \frac{(\beta_t - \gamma)^2}{2} . ka_2}$, thus:

$$\tau a_t = -[(\beta_t - \gamma) . ka_1 + \frac{(\beta_t - \gamma)^2}{2} . ka_2] \quad (27)$$

Let us observe that:

$$\tau a_t + \nu a_t = \gamma . ka_1 - \frac{\gamma^2}{2} . ka_2 + \beta_t . \gamma . ka_2 \quad (28)$$

while (26) and (27) are polynomials of degree 2 in β_t, (28) is a polynomial of degree 1 in β_t.

10. Let us now consider the random variable z_b encompassing rare disasters.

We refer to Barro (2011, page 2):

> "*The probability of a disaster is the constant* $p \geq 0$ *per unit of time. In a disaster, output contracts by the fraction* b *where* $0 < b \leq 1$" (page 2).

Then the factor e^{z_b} (see 23.2) is given by:

$$e^{z_b} = (1 - b) \quad (29)$$

With probability $(1 - p)$, no disaster occurs and therefore the factor e^{z_b} is given by:

$$e^{z_b} = 1 \quad (30)$$

Thus $\nu_b(\beta_t) = ln\, ln\, Ee^{z_b}$, with: $Ee^{\beta_t . z_b} = p . (1 - b)^{\beta_t} + (1 - p) . 1^{\beta_t}$, which can be written:

$$\nu_b(\beta_t) = ln\ ln\ [1 - p + p.(1-b)^{\beta_t}] \tag{31}$$

According to Barro (page7): "We *work with the transformed disaster size* $\xi = 1/(1-b)$*". For the threshold* b_0*,* $\xi_0 = 1/(1-b_0)$*. "As b approaches 1,* ξ *approaches* $+\infty$*, a limiting property that accords with the usual setting for a power-law distribution. We start with a familiar, single power law, which specifies the density function as* $f(\xi) = Q.\xi^{-(\alpha+1)}$ *for* $\xi \geq \xi_0$*, where* $Q > 0$ *and* $\alpha > 0$*. The condition that the density integrate to 1 from* ξ_0 *to* $+\infty$ *implies:*

$$Q = \alpha.\xi_0^{\alpha} \tag{32}$$

The variable ξ thus follows a Pareto probability distribution (*fat tail*). Observe that $(1-b) = \xi^{-1}$ and $(1-b)^{\beta_t} = \xi^{-\beta_t}$. Then (31) gives (see Appendix):

$$Ee^{\beta_t.z_b} = 1 - p + p.\frac{\alpha}{\beta_t+\alpha}.\xi_0^{-\beta_t} \tag{33}$$

On the same way:

$$Ee^{(\beta_{t-\gamma}).zb} = 1 - p + p.\frac{\alpha}{\beta_t-\gamma+\alpha}.\xi_0^{-(\beta_t-\gamma)} \tag{34}$$

Observe that (34) requests:

$$\beta_{t\_} - \gamma + \alpha > 0 \Leftrightarrow \beta_{t\_} > -(\alpha - \gamma) \tag{35}$$

Let us denote[6]

$$\varepsilon_0 = ln\ ln\ \xi_0 \tag{36}$$

From (33) and (34) we get:

$$\nu b(\beta_t) = ln\ (1 - p + p.\frac{\alpha}{\beta_t+\alpha}e^{-\beta_t.\varepsilon_0} \tag{37}$$

$$\tau b(\beta) = -ln\ (1 - p + p.\frac{\alpha}{\beta_t-\gamma+\alpha}.e^{-(\beta_t-\gamma).\varepsilon_0}) \tag{38}$$

$$\tau b(\beta) + \nu b(\beta_t) = -ln\ (1 - p + p.\frac{\alpha}{\beta_t-\gamma+\alpha}.e^{-(\beta_t-\gamma).\varepsilon_0}) + ln\ \ 1 - p + p.\frac{\alpha}{\beta_t+\alpha}e^{-\beta_t.\varepsilon_0} \tag{39}$$

This rare-risk component $\tau b(\beta) + \nu b(\beta_t)$ of the discount rate ρ_t is nonlinear in β_t, contrary (see (28)) to the gaussian component $\tau a(\beta) + \nu a(\beta_t)$.

We also will need to know the mathematical expectation kb_1 and the variance kb_2 of the random variable zb. They are given by (7):

$$kb_1 = -(\varepsilon_0 + \frac{1}{\alpha}).p \tag{40}$$

$$kb_2 = (\varepsilon_0^2 + 2.\frac{\varepsilon_0}{\alpha} + 2.\frac{1}{\alpha^2}).p - \ kb_1^2 \tag{41}$$

Part 2: An Application

The above model has been used to provide guidance to the Committee in its deliberations to revise the discount rate likely to be applied in France for the socio-economic evaluations of public investments, over the period from the reference year to the year 2070.

11. As regards the expectation of the annual rate of evolution z of the per capita GDP, the Committee has decided to adopt the "medium-low" macroeconomic scenario of the COR (*Conseil d'Orientation des Retraites*), whose version published in 2020 leads to a mean of

$$k_1 = 1.15\%\, p.a. \tag{42}$$

by 2070.

For the variance of z, the Committee has used the long series, since 1820, of GDP per capita in France, in purchasing power parity (PPP), drawn up by Bergeaud, Cette and Lecat (Banque de France). The value of the variance observed is 0.374% for the complete period starting in 1820, but it reaches 0.475% for the period starting in 1913, just before WW1: it is this value

$$k_2 = 0.475\% \tag{43}$$

that the Committee has taken into consideration.

12. As z is the sum of the random variables za and zb which are independent, we know that the mathematical expectations (cumulants number 1) are additive, as well as the variances (cumulants number 2):

$$k_1 = ka_1 + kb_1 \tag{44}$$

$$k_2 = ka_2 + kb_2 \tag{45}$$

If we assume that it is possible to estimate the cumulants kb_1 and kb_2 of the random component zb encompassing rare disasters, it will then be possible to determine by difference the cumulants ka_1 and $ka2$ relating to the gaussian random component za:

$$ka_1 = k_1 - kb_1 \tag{46}$$

$$ka_2 = k_2 - kb_2 \tag{47}$$

To that aim, the Committee has adopted the estimations made in the abovementioned article by Barro, who has gathered data over a period of 100 years in 36 countries. These parameters[7] are as follows:

$$\varepsilon_0 = 0.10\ ;\ \alpha = 6.86;\ \text{p}= 0.0383 \text{ (every 26 years on average)} \tag{48}$$

So that (40) and (41) deliver:

$$kb_1 = -0.94\% \text{ and } kb_2 = 0.304\% \tag{49}$$

Finally, (46) and (47) deliver the parameters of the gaussian component:

$$ka_1 = 2.09\% \text{ and } ka_2 = 0.171\% \tag{50}$$

13. To implement the model, we still need to specify the parameters of the utility function. The table hereafter quotes some estimations.

TABLE 1
REFERENCES

	δ	γ
	Pure time preference	Risk Aversion
	% par an	sans dimension
Cline (1992)	0	1.5
Arrow (1999)	0	2
Rapport Lebègue (2005)	1	2
Stern (2007)	0.1	1
Arrow (2007)		2 à 3
Dasgupta (2007)		2 à 4
Weitzman (2007)	2	2
Nordhaus(2008)	1.5	2
Barro (2006. 2011)	3	2.75 à 4.33
Gollier (2011)		2
Martin (2010)	3	
Pindyck(2013)		1 à 3
Rapport Quinet E (2013)	1	2
Green Book (UK) (2018)	0.5 +1	1

Source: Ch. Gollier

As regards the risk aversion coefficient γ, the Committee has preferred to raise its rounded value to 2.5, instead of 2 in Gollier (2011) and Quinet (2013).

Regarding the δ rate of pure time preference, it has been considered to be within a range of 0.4 to 0.6% p.a. As it is purely additive in the discount rate, it will be specified further.

14. Simulation in function of the elasticity β

Hereafter, to simplify, we write β, τ, ν, ρ instead of $\beta_t, \tau_t, \nu_t, \rho_t$

Let us consider for instance that β varies between -1 and γ (taken equal to 2.5).

For the sake of simplicity, let's provisionally assume that the purely additive parameter δ is zero and let us give the other model parameters the values already shown above:

$$\gamma = 2.5\ ;\ k_1 = 1.15\ \%\ p.a.;\ k_2 = 0.475\ \%\ p.a\ ;\ p = 0.0383\ ;\ \epsilon_0 = 0.10;\ \alpha = 6.86 \tag{51}$$

The analytical outcomes of the model "with rare disasters" are given in Table 2.

TABLE 2
WITH RARE DISASTERS, ANALYTICAL OUTCOMES

β	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0	2.5
τ *with*	-0.68	0.33	0.88	1.11	1.09	0.88	0.49	0.00
ν *with*	-0.89	-0.51	0.00	0.63	1.37	2.20	3.12	4.07
ρ *with*	-1.57	-0.19	0.88	1.75	2.47	3.08	3.61	4.07

in % p.a.

15. Comparison with the pure gaussian case, i.e. without rare disaster.

Let us now use the same model "without any rare disaster", which supposes that the parameter p is here zero. The outcomes are given in table 3:

TABLE 3
"WITHOUT RARE DISASTERS (PURE GAUSSIAN SCENARIO)"

β	-1.0	-0.5	0	0.5	1.0	1.5	2.0	2.5
τ *without*	1.14	1.33	1.40	1.35	1.18	0.90	0.50	0.00
ν *without*	-0.91	-0.52	0.00	0.64	1.39	2.26	3.25	4.31
ρ *without*	0.22	0.81	1.40	1.99	2.57	3.16	3.75	4.31

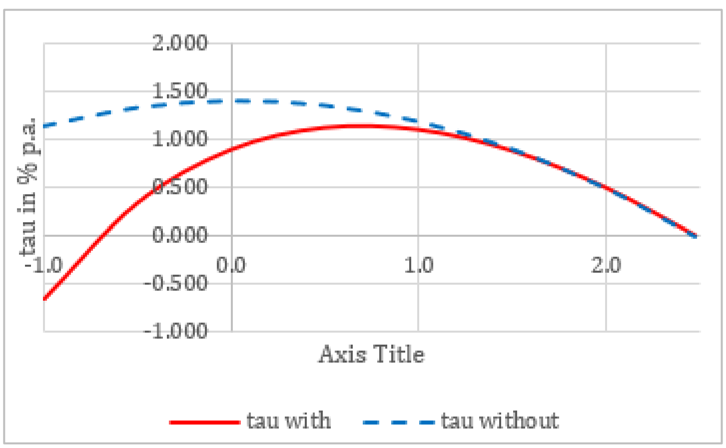
in % p.a.

It appears that the inclusion of rare disasters therefore has a significant impact on rates τ, ν and their sum ρ (here $\delta = 0$).

Let us pay special attention to the τ rate, which takes place in the fundamental definition (13) of the $SE\ NPV_t$ and could be called the "overall discount rate". It synthesizes the role of β, which intervenes separately in the ν rate and in the ρ discount rate.

Figure 1 shows the τ rate curve "with" and "without" rare disasters. The effect of taking rare disasters into account is nil if $\beta = \gamma$, but becomes all the more significant the lower β is below γ.

FIGURE 1
TAU WITH VERSUS WITHOUT RARE DISASTERS IN FUNCTION OF BETA



16. Simplifying formulas

The results of the model could be detailed and published in the form of tables. However, the Committee has wished to make available to the project sponsors simpler formulations, similar to those used in the pure gaussian case: i.e. a polynomial of degree 1 in β for the discount rate and a polynomial of degree 2 in β for the ν rate concerning the mathematical expectation of the benefit EA_t.

Approximations were then sought to achieve these objectives, while minimizing the errors introduced with respect to the aforementioned analytical results.

The priority has been devoted to the "overall discount rate" τ, which takes place in the direct definition of SE NPV. In order to fulfil its analytical property of being nil for $\beta = \gamma$, it has been approximated by a polynomial of degree 2 in $(\beta - \gamma)$:

$$\tau p(\beta) = -[u.(\beta - \gamma) + w.\frac{(\beta-\gamma)^2}{2}] \tag{52}$$

The numerical estimation of the parameters by the least-squares method has given:

$$u = 1.47\ ;\ w = 0.93 \tag{53.1}$$

(52) therefore gives:

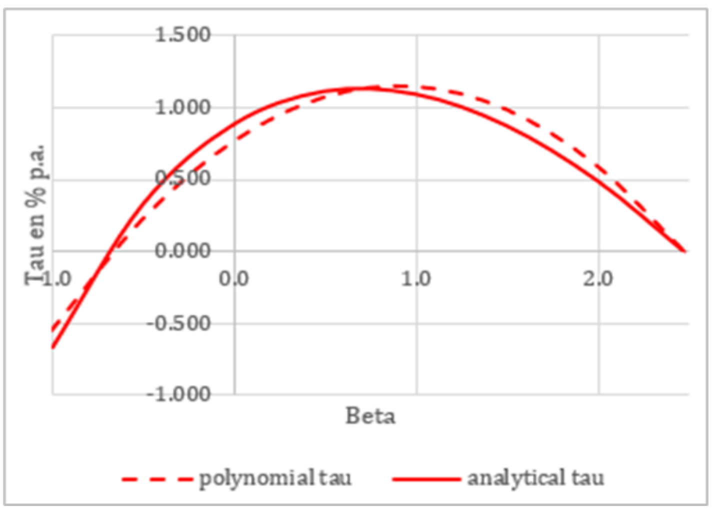
$$\tau p(\beta) = -[(\beta - \gamma) * 1.47 + \frac{(\beta-\gamma)^2}{2} * 0.93] \tag{53.2}$$

Other form of (53.2), as $\gamma = 2.5$:

$$\tau p(\beta) = 0.77 + \beta * 0.85 - \frac{\beta^2}{2} * 0.93 \tag{53.3}$$

Figure 2 shows the approximation thus introduced for the overall discount rate τ.

FIGURE 2
POLYNOMIAL TAU VERSUS ANALYTICAL TAU



The issue was then to share out the errors among the polynomials νp and ρp, having in mind that:

$$\textit{they are linked by the relationship } \rho p = \tau p + \nu p + \delta \tag{54}$$

$$\textit{and that } \rho p \textit{ should be a polynomial of degree } 1 \textit{ in } \beta. \tag{55}$$

Considering that ν should fulfil its analytical property to be nil for $\beta = 0$, we have approximated ν by a polynomial of degree 2 in β:

$$\nu p(\beta) = u'.\beta + w'.\frac{\beta^2}{2} \tag{56}$$

In order to verify the conditions (54) and (55), it appears necessary that $w' = w$, *i.e.* $w' = 0.93$. It is still needed to choose the parameter u'.

For that, it has been referred to a property of the analytical function $\nu(\beta)$, the derivative of which (see Appendix) is equal to k_1 if $\beta = 0$. Adopting this option[8] gives:

$$u' = k1 \tag{57}$$

This leads for the discount rate ρ to an approximation by the polynomial of degree 1:

$$\rho p(\beta) = rfp + \phi p.\beta \tag{58}$$

where:

$$rfp = \left(u.\gamma - w.\frac{\gamma^2}{2}\right) + \delta \tag{59}$$

$$\phi p = k_1 - (u - w.\gamma) \tag{60}$$

which gives

$$rfp = 0.77 + \delta\,;\ \phi = 2.00 \tag{61}$$

The outcomes of these approximations are given in table 4 (with provisionally $\delta = 0$)

TABLE 4
"WITH RARE DISASTERS. POLYNOMIAL APPROXIMATION"

β	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0	2.5
τ *polynomial*	-0.55	0.22	0.77	1.07	1.15	0.99	0.59	0.00
ν *polynomial*	-0.69	-0.46	0.00	0.69	1.62	2.78	4.17	5.72
ρ *polynomial*	-1.24	-0.24	0.77	1.77	2.77	3.77	4.77	5.72

in % p.a.

Comparing table 4 with table 2 provides information on the deviations introduced by the polynomial approximation.

17. To summarize:

Let us consider, at the margin of the GDP, a small project, with an advantage (benefits minus costs) in year t, denoted A_t, which is related to the per capita GPD Y_t by the relationship:

$$A_t = \bar{A}_t.Y_t^{\beta_t} \tag{11}$$

where β_t is the elasticity coefficient and $\bar{A}_t$ a scale factor specific of the project.

With the above-mentioned assumptions and notations, the rate ν_t gives the expectation EA_t through the formula:

$$EA_t = \bar{A}_t.e^{\nu_t.t} \tag{20}$$

the discount rate ρ_t gives the contribution of EA_t to the socioeconomic net present value of the projects:

$$SE\ NPV_t = e^{-\rho_t.t}.EA_t \tag{21}$$

Let us consider now that the pure time rate δ, which was provisionally supposed nil, be calibrated to the value $\delta = 0.43$ To simplify writing, let's omit the time index t.

"With rare disasters", the formulas obtained according to the polynomial approximation are:

$$\nu = \beta * 1.15 + \frac{\beta^2}{2} * 0.93 \quad \text{\% p.a.}$$
$$\rho = 1.20 + \beta * 2.00 \quad \text{\% p.a} \quad \text{with } \delta = 0.43 \tag{62}$$

"Without rare disasters (pure gaussian scenario)", the corresponding formulas obtained are[9]

$$\nu = \beta * 1.15 + \frac{\beta^2}{2} * 0.48 \quad \text{\% p.a.}$$
$$\rho = 1.83 + \beta * 1.18 \quad \text{\% p.a.} \quad \text{with } \delta = 0.43 \tag{63}$$

Table 5 shows the differences between the cases "with" (table 3) and "without rare disasters (pure gaussian scenario)" (Table 3), in function of β.

TABLE 5
DIFFERENCES BETWEEN THE CASES "WITH (POLYNOMIAL)" AND "WITHOUT (PURE GAUSSIAN SCENARIO)"

β	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0	2.5
Delta τ	-1.69	-1.10	-0.63	-0.28	-0.04	0.09	0.10	0.00
Delta ν	0.23	0.06	0.00	0.06	0.23	0.52	0.92	1.41
Delta ρ	-1.46	-1.04	-0.63	-0.22	0.19	0.60	1.02	1.41

in % p.a.

Taking account of rare disasters has thus a significant impact on the rates ν and ρ. Consequently, the impact on the socioeconomic net present value of the project (see Delta τ) is all the more noticeable the lower β is below γ.

ENDNOTES

1. Conseil d'Orientation des Retraites. Scenario medium-low.
2. Constant relative risk aversion.
3. Supposed constant over time.
4. These parameters may depend on the nature of the advantage, which may differ from one year to another one: e.g. investment year versus operating year, etc...
5. The function $ln\, ln\, Ee^{x..z}$ is the generating function of the cumulants of the random variable za.
6. It could be written: $z_{b0} = -\varepsilon_0$
7. Or considering the moment function: $\mathrm{h(w)} = 1 - \mathrm{p} + p.Q.\int_{\varepsilon_0}^{+\infty} e^{-w.\epsilon}.e^{-\alpha.\varepsilon}.d\varepsilon = 1 - p + p.\frac{\alpha}{w+\alpha}.e^{-w.\varepsilon_0}$
   i.e.: $\mathrm{h(w)} = 1 - p + p.(1 - \frac{1}{\alpha}.\mathrm{w} + \frac{1}{\alpha^2}w^2 + \cdots).\left(1 - w.\varepsilon_0 + \frac{1}{2}.w^2.\varepsilon_0^2 + \cdots\right)$
   I.e.: $\mathrm{h(w)} = 1 - (\varepsilon_0 + \frac{1}{\alpha}).p.w + \frac{1}{2}.(\varepsilon_0^2 + 2.\frac{\varepsilon_0}{\alpha} + 2.\frac{1}{\alpha^2}).p.w^2) + \cdots$
8. The threshold of rare disasters estimated by Barro (2011) is: $b_0 = 9{,}5\%$; which gives: $\epsilon_0 = -ln\, ln\,(1 - 0.095) = 0.0998$, rounded to 0.10
9. Other options were envisaged, like estimating u' by the least square method.
10. Verification: let us consider a pure gaussian variable with: k_1=1,15% p.a. ; $k_2 = 0{,}475\%\, p.a.$
    Let us transpose the formula (26): $\nu = \beta * 1.15 * + \frac{\beta^2}{2} * 0.475$
    With $\delta = 0$, let us transpose (28): $\rho = 1.15 * 2.5 - 0.5 * 2.5^2 * 0.475 + \beta * 2.5 * 0.475$ = 1,40+$\beta * 1{,}18$
    And adding $\delta = 0.43$: ρ = 1.83+$\beta * 1.18$

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APPENDIX

1. (31) gives (assuming $\beta_t + \alpha > 0$):

   $$Ee^{\beta_t . z_b} = 1 - p + p.Q.\int_{\xi_0}^{+\infty} \xi^{-\beta_t}.\xi^{-(\alpha+1)}.d\xi = 1 - p + p.Q.\left[\frac{\xi^{-(\beta_t+\alpha)}}{-(\beta_t+\alpha)}\right]_{\xi_0}^{+\infty} = 1 - p + p.Q.\frac{\xi_0^{-(\beta_t+\alpha)}}{\beta_t+\alpha}$$

   As $Q = \alpha.\xi_0^{\alpha}$:

   $$Ee^{\beta_t . z_b} = 1 - p + p.\frac{\alpha}{\beta_t+\alpha}.\xi_0{}^{-\beta_t}$$

   Finally:

   $$v_b = \ln[1 - p + p.\frac{\alpha}{\beta_t+\alpha}e^{-\beta_t.\varepsilon_0}] \text{where} \quad \varepsilon_0 = \ln \xi_0$$

2. We also will need to know the mathematical expectation kb_1 and the variance kb_2 oh the random variable z_b.

Observe that: $z_b = (1-b) = \xi^{-1} = e^{-\ln\xi}$,
Let us denote: $\varepsilon = \ln\xi$
Thus: $z_b = -\varepsilon$ and $\xi = e^{\varepsilon}$, hence $\frac{d\xi}{d\varepsilon} = e^{\varepsilon}$
The probability density of ε is then:

$$l(\varepsilon) = f(\xi).\frac{d\xi}{d\varepsilon} = Q.\xi^{-(\alpha+1)}.e^{\varepsilon} = Q.e^{-(\alpha+1).\varepsilon}.e^{\varepsilon} = Q.e^{-\alpha.\varepsilon}$$

Let us consider the non-centered statistical moments of z_b: $mb_n = p.Q.\int_{\varepsilon_0}^{+\infty}(-\varepsilon)^n.e^{-\alpha.\varepsilon}.d\varepsilon$
Integrating[1] by parts, it comes:

$$mb_1 = -(\varepsilon_0 + \frac{1}{\alpha}).p$$

$$mb_2 = \frac{1}{2}.(\varepsilon_0^2 + 2.\frac{\varepsilon_0}{\alpha} + 2.\frac{1}{\alpha^2}).p$$

Then:

$$kb_1 = mb_1$$

$$kb_2 = mb_2 - mb_1^2$$

3. Proposition:

$$\left[\frac{dv(\beta)}{d\beta}\right]_{\beta=0} = k_1 \quad (320)$$

Proof

Reminder: $v(\beta) = \ln Ee^{\beta..z}$. Then: $\frac{dv(\beta)}{d\beta} = \frac{d(Ee^{\beta..z})/d\beta}{Ee^{\beta..z}} = \frac{E(z.e^{\beta..z})}{Ee^{\beta..z}}$ (320.1)

Let us transcribe (320.1) for the variable: $\frac{dva(\beta)}{d\beta} = \frac{d(Ee^{\beta..za})/d\beta}{Ee^{\beta..za}} = \frac{E(za.e^{\beta..za})}{Ee^{\beta..za}}$
When β tends towards 0, then $Ee^{\beta..za}$ tends towards 1 and $E(za.e^{\beta..za})$ tends towards Eza, which is the mathematical expectation of za, denoted ka_1. Finally:

$$\left[\frac{dva(\beta)}{d\beta}\right]_{\beta=0} = ka_1 \quad (320.2)$$

On the same way, let us transcribe (320.1) for the variable zb. It comes:

$$\left[\frac{dvb(\beta)}{d\beta}\right]_{\beta=0} = kb_1 \quad (320.3)$$

The variable z being the sum of the independent variables za et zb:

$$\left[\frac{dv_{a+b}(\beta)}{d\beta}\right]_{\beta=0} = \left[\frac{dva(\beta)}{d\beta}\right]_{\beta=0} + \left[\frac{dvb(\beta)}{d\beta}\right]_{\beta=0} = ka_1 + kb_1 = k_1 \quad (320)$$