Supply Chain Inventory Optimization With Multiple Types of Customers and Equal Priorities: A Study Using Heuristic and Simulation Methods

Bo Li
California State University, Los Angeles

In this study, the optimal inventory policies for a supply chain with multiple types of customers is investigated to minimize the total inventory cost. Each customer type has specific demand patterns in terms of ordering frequency and order quantity; moreover, there is no difference in priority among the customers. Due to the computational complexity of this optimization problem, we develop a reduced form of the original problem and propose a heuristic that is easily implementable by practitioners. In addition, a simulation model is developed on the same problem setting as a benchmark. To illustrate the implementation and the effectiveness of the proposed heuristic, a numerical example is studied, followed by a sensitivity analysis of the key parameters. The numerical results show that our heuristic provides a comparable performance as simulation-based optimization (a 2.6% difference), and the sensitivity analysis shows a reliable robustness of the model (less than 5.2% difference).

Keywords: supply chain management, inventory optimization, multiple customer types, heuristic, simulation

INTRODUCTION

Inventory management is a critical practice for most companies, regardless of the history, scope and size of an organization (Gills, Thomas, McMurtrey, & Chen, 2020; Qin, Wang, Vakharia, Chen, & Seref, 2011). Supply chain inventory management is even more challenging due to the complexity and dynamics of networks of multiple organizations (Konstantaras, Skouri, & Lagodimos, 2019; Kubat & Wang, 2020; Mwangola, 2018). In this study, we investigate supply chains with two types of customers, namely, commercial customers and retail customers, who have different ordering behaviors. Commercial customers order less frequently but with a larger average order size, while retail customers order more frequently but with a smaller average order size. Real-world examples can be observed in many industries, such as the grocery, pharmacy, and construction industries.

Veinott (1965) first studied inventory management for different types of customers with different priorities, and the demand from high-priority customers should be satisfied before the demand from low-priority customers. Topkis (1968) extended this work to establish a set of critical inventory levels that trigger a decision to stop deliveries to certain types of customers. Sobel and Zhang (2001) consider a periodic review inventory system with deterministic (scheduled) demand and stochastic (unscheduled) demand. The deterministic demand must be satisfied immediately, and the stochastic demand may be backordered. More recently, ElHafsi, Fang, and Hamouda (2021) proposed heuristics to produce the inventory management policy of a system where a manufacturer serves two types of customers: One type has long-term commitment and backorder, and the other has short-term commitment and lost sales.

In contrast to the literature, in this study we assume a more comprehensive and broader definition of multiple types of customers who have several different characteristics, such as order frequency, order size, and shortage costs. In addition, to be more practical, the requirements of customer priority difference in the traditional inventory rationing mechanism are removed. The objective function is to minimize the total inventory cost, which is the sum of ordering cost, holding cost, and shortage cost. Because of the computational complexity of this optimization problem, we reduce the original problem to a solvable form and develop a heuristic to determine the optimal inventory policies.

The rest of the paper is organized as follows: In Section 2, we define the problem of this study with notation and mathematical formation, and in Section 3, the heuristic is developed based on a reduced form of the original optimization problem. A simulation model is built in Section 4 as a benchmark for comparison with the heuristic method. In Section 5, a numerical study is presented to demonstrate the implementation and effectiveness of the proposed heuristic, followed by a sensitivity analysis of the key parameters. Finally, conclusions and suggestions for future research are discussed in Section 6.

DEFINITION OF THE OPTIMIZATION PROBLEM

We first introduce the notation used in modeling the inventory system, and then, we define the mathematical form of the optimization problem.

Notation Used in the Optimization Problem

The system investigated in this study involves two types of customers with different characteristics. Each customer type has a demand following a compound Poisson distribution with Poisson arrival rates of λ_1 and λ_2 and an exponential order quantity with means of μ_1 and μ_2 units per order, respectively. No partial shipment is allowed, and any unmet demand is lost (no backlog), which leads to a shortage cost, G_1 or G_2, depending on the type of customer. The problem is to find an optimal continuous review policy for order quantity (Q) and reorder level (r) that minimizes the total inventory cost per unit time. The notation used to describe the critical data of this problem are summarized in Table 1.

TABLE 1
NOTATION USED IN THE OPTIMIZATION PROBLEM

Notation	Meanings
λ_1 , λ_2	Means of arrival rate (for commercial and retail customers, respectively)
μ_1 , μ_2	Means of customer order size
L	Lead time of replenishment from external suppliers (fixed)
D_1, D_2	Means of the demands per unit time
D	Means of the total demand per unit time ($D = D_1 + D_2$)
N_{s1}, N_{s2}	Expected numbers of unfulfilled customer orders in one cycle
G_1, G_2	Shortage costs per unfulfilled order
S_1, S_2	Average units of shortage amount per cycle
S	Average units of total shortages occurring each cycle. $S = S_1+S_2$
Q	Order quantity
r	Reorder level
A	Ordering cost for placing an order
H	Holding cost
$E[C(Q, r)]$	Expected total cost

The Objective Function

The objective of this optimization is to minimize the expected total inventory cost $E[C(Q,r)]$ per unit time. The objective function contains three components as follows, and each of the three components is also defined.

$$E[C(Q,r)] = Expected\ ordering\ cost + Expected\ holding\ cost + Expected\ shortage\ cost$$

For the lost-sales case, Rosling (2002) has shown that the expected duration of a cycle can be approximated by$(Q+S)/D$ rather than Q/D, which is used in the case of problems with back ordering. Thus, the ordering cost is as follows.

$$Expected\ ordering\ cost\ =\ A\frac{D}{Q+S}$$

In the lost sales case, Johnson and Montgomery (1974) suggests the following expression to approximate the holding cost.

$$Expected\ holding\ cost = H\left(r - LD + S + \frac{Q}{2}\right)$$

The shortage cost is defined as the cost incurred per stockout, and its value depends on the type of customer. We use the notions of G_1 and G_2 for commercial and retail customers, respectively. The expected numbers of unfulfilled orders, N_{S1} and N_{S2}, are given by the ratio of the units of total shortage and the average units of each customer order. According to Rosling (2002), we use $(Q+S)/D$ to represent the expected duration of a cycle, and the expected shortage cost is expressed as follows.

$$Expected\ shortage\ cost\ = \left(G_1\frac{S_1}{\mu_1} + G_2\frac{S_2}{\mu_2}\right)\frac{D}{Q+S}$$

Based on the above analysis, we thus obtain the expected total costs per unit time as follows.

$$E[C(Q,r)] = A\frac{D}{Q+S} + H\left(r - LD + S + \frac{Q}{2}\right) + \left(G_1\frac{S_1}{\mu_1} + G_2\frac{S_2}{\mu_2}\right)\frac{D}{Q+S} \quad (1)$$

where $S = S_1 + S_2$, $\lambda = \lambda_1 + \lambda_2$, $D = D_1 + D_2$, $D_1 = \lambda_1\mu_1$, and$D_2 = \lambda_2\mu_2$

HEURISTIC DEVELOPMENT

In this section, we first derive a reduced form of the original NP hard optimization problem, and then, a heuristic is developed based on this reduced form, which requires much less effort to solve in practice.

A Reduced Form of the Objective Function

All the variables in Equation (1) are expressed in terms of Q and r, except the average units of shortage amount per period, S_1 and S_2. To express S_1 and S_2 by using Q and r only, we conceptualize the reordering point r as the combination of two parts (r_1 and r_2, $r = r_1 + r_2$) where r_1 and r_2 represent the reordering levels for the two types of customers. We further analyze r_1 and r_2 by using the weight of each customer type, and express them as $r_1 = w_1 r$, and $r_2 = w_2 r$, where $w_1 + w_2 = 1$, and $w_1 = \frac{\lambda_1\mu_1}{\lambda_1\mu_1+\lambda_2\mu_2}$, $w_2 = \frac{\lambda_2\mu_2}{\lambda_1\mu_1+\lambda_2\mu_2}$. Here, the weight, w_1, presents the ratio of the expected lead-time demand for commercial customers to the total expected lead-time demand of both customer types. A similar logic applies to w_2.

We thus obtain the expressions of shortage units from commercial customers and retail customers, S_1 and S_2, as $S_1 = \int_{r_1}^{\infty}(x - r_1)f_1(x)dx$ and $S_2 = \int_{r_2}^{\infty}(x - r_2)f_2(x)dx$, where $f_1(x)$ and $f_2(x)$ are the density functions of lead time demand from commercial and retailer customers, respectively. Assuming that the lead time demand follows an exponential distribution, $f_1(x)$ and $f_2(x)$ can be presented as $f_1(x) = \frac{1}{L\lambda_1\mu_1}e^{-\frac{x}{L\lambda_1\mu_1}}$ and $f_2(x) = \frac{1}{L\lambda_2\mu_2}e^{-\frac{x}{L\lambda_2\mu_2}}$, respectively. The expected values of the lead-time demands for the two customer types are$L\lambda_1\mu_1$ and $L\lambda_2\mu_2$, respectively.

Lemma 1: According to the above definitions, the shortage costs for commercial and retailer customers are $S_1 = L\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}}$, $S_2 = L\lambda_2\mu_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}$, and $S = L\left(\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}} + \lambda_2\mu_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}\right)$. The proof is provided in Appendix 1.

Based on the above analysis and the results of Lamma 1, objective function (1), which includes only the variables Q and r, can be written as follows:

$$E[C(Q,r)] = H\left(r - LD + L\left(\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}} + \lambda_2\mu_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}\right) + \frac{Q}{2}\right) + \left(A + G_1 L\lambda_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}} + G_2 L \times \lambda_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}\right)\frac{D}{Q + L\left(\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}} + \lambda_2\mu_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}\right)} \quad (2)$$

Heuristic of the Optimal Solution

To find the optimal solution to minimize $E[C(Q,r)]$, we calculated the partial derivatives of the total inventory cost with respect to variables Q and r, and then, we set them equal to zero to identify the potential solutions. The results are presented in Lemmas 2 and 3.

Lemma 2: The partial derivatives of $E[C(Q,r)]$ with respect to Q and r are $\frac{\partial E[C(Q,r)]}{\partial Q} = \frac{1}{2}H - \frac{AD + (G_1\lambda_1 + G_2\lambda_2)LZ_r D}{(Q + LDZ_r)^2}$ and $\frac{\partial E[C(Q,r)]}{\partial r} = H(1 - \mathrm{Z_r}) - \frac{(G_1\lambda_1 + G_2\lambda_2)\,\mathrm{Z_r}}{Q + LD\mathrm{Z_r}} + \frac{(A + G_1 L\lambda_1 \mathrm{Z_r} + G_2 L\lambda_2 \mathrm{Z_r})\,D\mathrm{Z_r}}{(Q + LD\mathrm{Z_r})^2}$, where $Z_r = e^{-\frac{r}{LD}}$. The proof of Lemma 2 is given in Appendix 2.

Lamma 3: To optimize the objective function, we let $\frac{\partial E(C(Q,r))}{\partial Q} = 0$ and $\frac{\partial E(C(Q,r))}{\partial r} = 0$, and we obtain the following system of equations. The proof of Lemma 3 is provided in Appendix 3.

$$H(L\mathrm{D})^2\mathrm{Z_r^2} + 2(H\mathrm{Q} - (G_1\lambda_1 + G_2\lambda_2))\mathrm{LDZ_r} + HQ^2 - 2AD = 0 \quad (3)$$

$$H(1 - \mathrm{Z_r})(Q + LD\mathrm{Z_r})^2 - \left(\frac{G_1 w_1 \mathrm{Z_r}}{\mu_1} + \frac{G_2 w_2 \mathrm{Z_r}}{\mu_2}\right) D(Q + LD\mathrm{Z_r}) + (A + G_1 L\lambda_1 \mathrm{Z_r} + G_2 L\lambda_2 \mathrm{Z_r})D\mathrm{Z_r} = 0 \quad (4)$$

Thus, the original optimization problem is reduced to solving the system of Equations (3) and (4) for (Q, r). The system of equations can be solved by using mainstream software, such as Maple and MATLAB. We illustrate the implementation of this heuristic later through a numerical study in Section 5.

DEVELOPMENT OF THE SIMULATION MODEL

In addition to the proposed heuristic model based on the expected values, we further develop a comparative model by using a simulation-based optimization approach that directly models random situations in practical scenarios. The results of this simulation model are compared with those of the heuristic model in a numerical study.

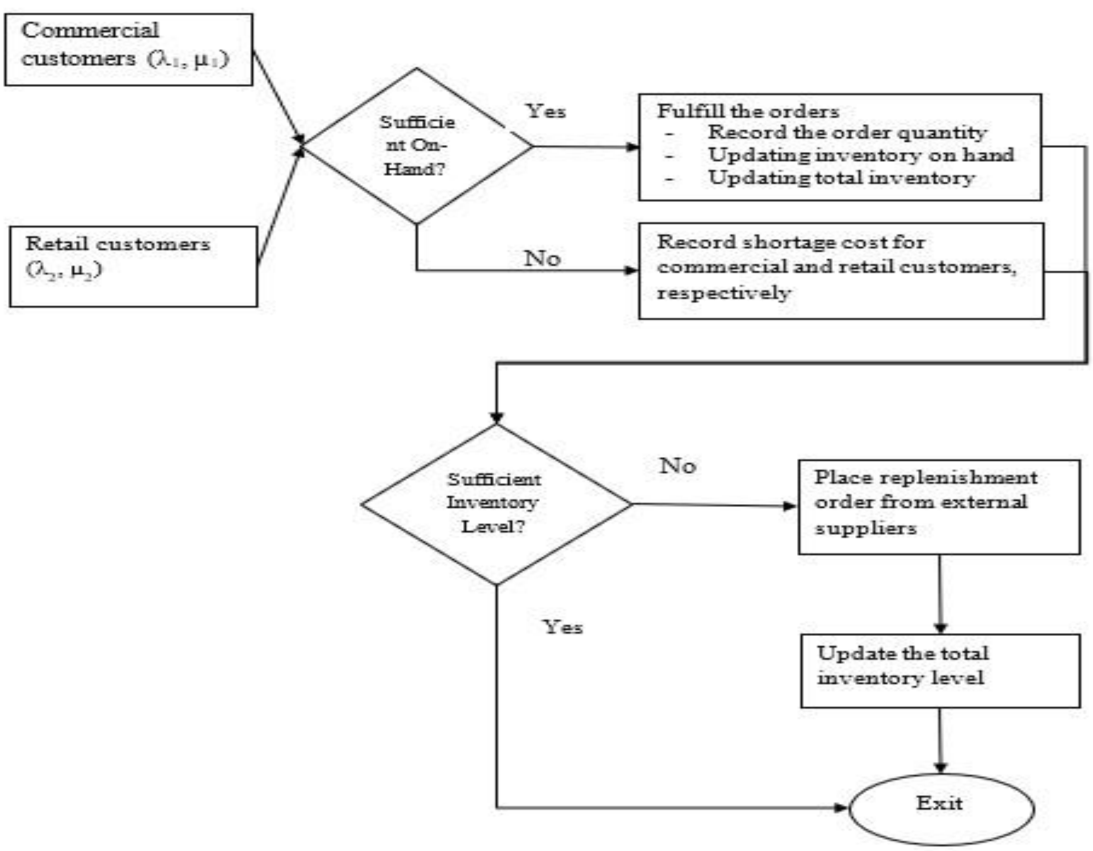
The logic flow of the simulation is presented in Fig. 1. The simulation system generates two types of entities to represent the two types of customers in the inventory problem. The entities (commercial and retail customers) enter the system following Poisson distributions with arrival rates λ1 and λ2, respectively. The order quantity of each customer is assigned following an exponential distribution with average order quantity, µ1 and µ2.

After a specific customer entity enters the simulation system, the simulation model first checks whether the currently available on-hand inventory is sufficient for this customer's order. If the answer is "Yes", then this specific order is fulfilled immediately from the available on-hand inventory. At the same time, the system updates both the on-hand inventory level and the total inventory level, which include both on-hand inventory and in-transit inventory. If the answer is "No", then a shortage occurs, and the cost is recorded according to the customer type.

Before the entity exits, the simulation system checks the most updated total inventory level to decide whether it is necessary to place an inventory replenishment order from the external supplier. On the one hand, if the current total inventory level is equal to or lower than the designed reordering point r, then a new replenishment order is placed from the external supplier. In this case, the total inventory level must be updated immediately, but the on-hand inventory waits for a period of lead time. Thus, the simulated entity enters a queue delay for a lead-time period before updating the on-hand inventory of the system by adding a newly available amount. On the other hand, if the current total inventory level is higher than the designed reordering point r, then the entity exits the system without making any replenishment orders.

Following this design logic, we implement the simulation model by using the ARENA@ software, which is one of the most widely used simulation software for discrete event modeling (Tsai, Wang, & Hung, 2023; Yousefi, Yousefi, & Fogliatto, 2020).

FIGURE 1
FLOWCHART OF THE SIMULATION MODEL DESIGN



A NUMERICAL STUDY

To illustrate the application and to evaluate the effectiveness of the heuristic, we solve a numerical example by using both the proposed heuristic model and the simulation-based optimization method. We first identify the best performance by using simulation-based optimization as a benchmark, and then, we apply the proposed heuristic to the same dataset. We thus investigate how closely the performance provided by the proposed heuristic compares with that of simulation-based optimization. To assess the robustness of the models, we further conduct a sensitivity analysis on the key parameters used in this numerical study.

Data Used in the Numerical Study

The data and parameters of the numerical example are listed in Table 2. The dataset follows the characteristics described in the previous sections: The two types of customers have different order sizes and shortage costs; commercial customers typically have larger order sizes but lower ordering frequencies and higher shortage costs; and retail customers typically have smaller order sizes but higher ordering frequencies and lower shortage costs.

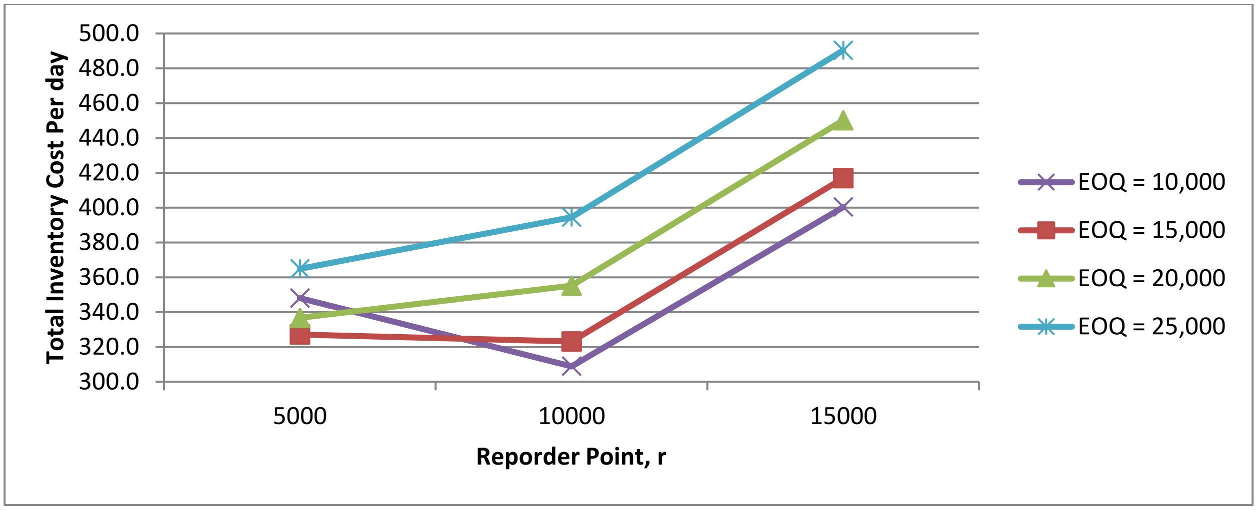
TABLE 2
PARAMETERS USED IN THE NUMERICAL EXAMPLE

Parameters	Values
Order frequency (orders/day) – commercial customers	Poisson (1)
Order frequency (orders/day) – retail customers	Poisson (4)
Order size (units/order) – commercial customers	Exp (700)
Order size (units/order) – retail customers	Exp (75)
Shortage cost ($/order) – commercial customers	1400
Shortage cost ($/order) – retail customers	150
Lead time (days)	5
Ordering cost ($)	1,000
Holding cost ($/unit/day)	0.02
Review policy	Periodic review
Length of simulation (hour)	80,000
Warm-up period (hour)	8,000
Replications	30

Experimental Results

By using the simulation model established in Section 4 and the data given in Table 2, we first search for the solution of the optimization problem based on the simulations of different combinations of Q and r. The results are summarized in Figure 2, and accordingly, we find that the optimal solution is Q = 10,000 and r = 10,000, which leads to a total inventory cost of $309 per day. It is also worth to noting that the second-best solution is Q = 15,000 and r = 10,000, which leads to a total cost of $323 per day.

FIGURE 2
TOTAL COST PER DAY FOR VARIOUS ORDER QUANTITIES (Q) AND REORDER LEVELS (r)



We next apply the proposed heuristic model to the same data and solve the system of Equations (3) and (4) by using Maple[@] software. We find that the optimal solution is Q = 14,934 and r = 9,647, which is very close to the second-best solution obtained from the simulation optimization. In addition, the corresponding total inventory cost is $317 per day, which is between the best result of $309 per day and the second-best result of $323 per day obtained by the simulation-based optimization.

Hence, the proposed heuristic provides a near-best solution and a comparable performance with simulations (only a 2.6% difference), but it requires much less effort and time to solve. From a practical standpoint, the proposed heuristic has significant advantages compared with the complex simulation optimization method.

Sensitivity Analysis of the Key Parameters

To further evaluate the robustness of the obtained optimal solution, we conduct sensitivity analysis on the following key parameters: 1) shortage cost for commercial customers, G_1; 2) shortage cost for retail customers, G_2; 3) holding cost per day per unit, H; and 4) ordering cost per order, A. For each parameter, we test the new values in the range of ±10% of the original given value.

Shortage Cost (±10%)

To assess the shortage cost parameters G_1 and G_2, we designed experiments involving different combinations, where G_1 was set to 1,540 (+10%) and 1,260 (-10%) and G_2 was set to 165 (+10%) and 135 (-10%). Under each of the new parameters, we first use the proposed heuristic to find the optimal solution $(Q; r)_N$ and its corresponding total inventory cost TIC_N. We then apply the original optimal solutions to the simulation model with the new parameters and obtain the total inventory costs TIC_H and TIC_S, where H and S represent heuristic and simulation, respectively. Finally, the results were compared for differences in percentages (Table 3).

TABLE 3
THE EXPERIMENTAL PERFORMANCE OF TOTAL INVENTORY COSTS FOR EXAMINING THE EFFECTS OF SHORTAGE COSTS (G_1 AND G_2)

	G_1 = 1,540		G_1 =1260	
	G_2 = 165	G_2 = 135	G_2 = 165	G_2 = 135
$(Q; r)_N$	(15,036;10,078)	(14,977; 9,824)	(14,887; 9,464)	(14,811; 9,176)
TIC_N	325.3	321.2	313.9	309.2
TIC_H	318.0	317.9	316.6	316.4
-- Difference in %	2.30%	1.04%	0.85%	2.28%
TIC_S	309.8	309.6	308.1	307.9
-- Difference in %	5.00%	3.75%	1.88%	0.42%

The results of the sensitivity analysis on G_1 and G_2 in Table 3 show that with ±10% variations in the shortage costs G_1 and G_2, the new total inventory cost is still in the range of 5% of the performance of the original optimal solutions. This demonstrates the robustness of the heuristic optimal solution on the parameters G_1 and G_2.

Holding Cost and Ordering Cost (±10%)

We continue to test the holding cost H and ordering cost A by changing them in the range of ±10%, where H is set as 2.2% and 1.8%, and A is set as 900 and 1,100. Similar to the analysis of G_1 and G_2, we find the optimal solution $(Q; r)_N$, TIC_N, TIC_H, and TIC_S with the new values of the parameters H and A, and then, we compare the values for the difference in percentage. The results are reported in Table 4.

TABLE 4
THE EXPERIMENTAL PERFORMANCE OF TOTAL INVENTORY COSTS FOR EXAMINING THE EFFECTS OF HOLDING COST (H) AND ORDERING COST (A)

	H = 2.2%		H = 1.8%	
	A = 1,100	A = 900	A = 1,100	A = 900
$(Q; r)_N$	(14,823; 9,220)	(14,031; 9,473)	(15,949; 9,853)	(15,046; 10,124)
TIC_N	341.2	326.8	306.5	293.4
TIC_H	348.2	334.9	299.6	286.3
-- Difference in %	2.01%	2.42%	2.30%	2.48%
TIC_S	338.8	319.0	298.7	278.9
-- Difference in %	0.71%	2.45%	2.61%	5.20%

According to the results of the sensitivity analysis presented in Table 4, we note that with ±10% variations, the performance is still in the range of 5.2% of the simulated optimal solution, and in most cases, the differences are in the range of 3%. The results demonstrate the stability of the heuristic-based optimal solution for parameters H and A.

CONCLUSIONS

In this study, we addressed the problem of finding the optimal inventory policies for a supply chain facing multiple types of customers and multiple periods of continuous demand with a compound Poisson distribution, and all the customers have equal priority. A heuristic was developed to determine the appropriate inventory policy that minimized the total cost, and a simulation model was used as a benchmark. In our numerical study, we illustrated the implementation of the proposed models and demonstrated the effectiveness of the heuristic by comparing the performance of the optimal solutions obtained by different models. The sensitivity analysis shows the robustness of the proposed models on the key parameters.

This work is an explorative study to develop a solvable heuristic for supply chain managers of an inventory system with multiple types of customers with equal priorities. Due to its explorative nature, there are many ways to extend this study to further research. In this study, we focused on two types of customers, and future research could generalize the results to systems with more than two customer types. In addition, we assume that the customer arrival rates follow Poisson distributions and that the order quantity of each customer follows exponential distributions. Future research could analyze other types of distributions to reflect practical situations in different business contexts and environments.

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APPENDIX 1: THE PROOF OF LEMMA 1

Next, we derive expressions for S_1 and S_2.

$$S_1 = \int_{r_1}^{\infty} (x - r_1) f_1(x) dx$$
$$= \int_{w_1 \times r}^{\infty} (x - w_1 r) \frac{1}{L\lambda_1\mu_1} e^{-\frac{x}{L\lambda_1\mu_1}} dx$$

Using the integration by parts, S_1 can be written as follows

$$S_1 = -\int_{w_1 r}^{\infty} (x - w_1 r) \times \left(e^{-\frac{x}{L\lambda_1\mu_1}}\right)' dx$$
$$= -(x - w_1 r) \times \left(e^{-\frac{x}{L\lambda_1\mu_1}}\right)\Big|_{x = w_1 \times r}^{x = \infty} + \int_{w_1 \times r}^{\infty} e^{-\frac{x}{L\lambda_1\mu_1}} \times (x - w_1 r)' dx$$

Here, $\lim_{x\to\infty} e^{-\frac{x}{L\lambda_1\mu_1}} \to 0$, and $\lim_{x\to\infty} (x - w_1 r) \to \infty$. Because exponential function has the higher order, $\lim_{x\to\infty} (x - w_1 r) * e^{-\frac{x}{L\lambda_1\mu_1}} \to 0$. After substitution and simplification, S_1 is written as follows.

$$S_1 = \int_{w_1 r}^{\infty} e^{-\frac{x}{L\lambda_1\mu_1}} dx$$
$$= -L\lambda_1\mu_1 e^{-\frac{x}{L\lambda_1\mu_1}}\Big|_{x = w_1 r}^{x = \infty}$$

$\lim_{x\to\infty} e^{-\frac{x}{L\lambda_1\mu_1}} \to 0$, thus,

$$S_1 = -\left(0 - L\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}}\right)$$
$$= L\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}}$$

Similarly, we can get S_2 and S (the average units of total shortages occur each period, $S = S_1+S_2$).

$$S_1 = L\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}}$$

$$S_2 = L\lambda_2\mu_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}$$

And $S = S_1 + S_2 = L\left(\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}} + \lambda_2\mu_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}\right)$

APPENDIX 2: PROOF OF LEMMA 2

$$\frac{\partial E\big(C(Q,r)\big)}{\partial Q} = -\frac{AD}{(Q+S)^2} + \frac{1}{2}H - \left(G_1\frac{S_1}{\mu_1} + G_2\frac{S_2}{\mu_2}\right)\frac{D}{(Q+S)^2}$$

$$= -\frac{A(\mu_1\lambda_1+\mu_2\lambda_2)}{\left(Q + L\mu_1\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + L\mu_2\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)^2} + \frac{1}{2}H - \frac{\left(G_1 L\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + G_2 L\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)(\mu_1\lambda_1+\mu_2\lambda_2)}{\left(Q + L\mu_1\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + L\mu_2\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)^2}$$

$$\frac{\partial E\big(C(Q,r)\big)}{\partial r} = A\frac{-D}{(Q+S)^2}\frac{\partial S}{\partial r} + H\left(1+\frac{\partial S}{\partial r}\right) + \frac{-D}{(Q+S)^2}\frac{\partial S}{\partial r}\left(G_1\frac{S_1}{\mu_1} + G_2\frac{S_2}{\mu_2}\right) + \frac{D}{Q+S}\left(\frac{G_1}{\mu_1}\frac{\partial S_1}{\partial r} + \frac{G_2}{\mu_2}\frac{\partial S_2}{\partial r}\right)$$

Where

$$\frac{\partial S_1}{\partial r} = \frac{\partial}{\partial r}\left(L\lambda_1\mu_1 e^{-\frac{w_1 r}{L\lambda_1\mu_1}}\right) = -w_1\, e^{-\frac{w_1 r}{L\lambda_1\mu_1}}$$

$$\frac{\partial S_2}{\partial r} = \frac{\partial}{\partial r}\left(L\lambda_2\mu_2 e^{-\frac{w_2 r}{L\lambda_2\mu_2}}\right) = -w_2\, e^{-\frac{w_2 r}{L\lambda_2\mu_2}}$$

$$\frac{\partial S}{\partial r} = -\left(w_1\, e^{-\frac{w_1 r}{L\lambda_1\mu_1}} + w_2\, e^{-\frac{w_2 r}{L\lambda_2\mu_2}}\right)$$

Substituting the above formulas into $\frac{\partial E(C(Q,r))}{\partial r}$, we have the following.

$$\frac{\partial E\big(C(Q,r)\big)}{\partial r} = H\left(1 - w_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} - w_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right) - \frac{\left(\frac{G_1 w_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}}}{\mu_1} + \frac{\left(G_2 w_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)}{\mu_2}\right)(\mu_1\lambda_1+\mu_2\lambda_2)}{Q + L\mu_1\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + L\mu_2\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}} + \frac{\left(A + G_1 L\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + G_2 L\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)(\mu_1\lambda_1+\mu_2\lambda_2)}{\left(Q + L\mu_1\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + L\mu_2\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)^2} \times \left(w_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + w_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)$$

For simplification, let us use E_r to represent an exponential as shown below.

$$E_r = e^{-\frac{r}{L}}$$

Then

$$\frac{\partial E\big(C(Q,r)\big)}{\partial Q} = -\frac{A(\mu_1\lambda_1+\mu_2\lambda_2)}{\left(Q+L\mu_1\lambda_1 E_r{}^{\frac{w_1}{\mu_1\lambda_1}}+L\mu_2\lambda_2 E_r{}^{\frac{w_2}{\mu_2\lambda_2}}\right)^2}+\frac{1}{2}H - \frac{\left(G_1L\lambda_1 E_r{}^{\frac{w_1}{\mu_1\lambda_1}}+G_2L\lambda_2 E_r{}^{\frac{w_2}{\mu_2\lambda_2}}\right)(\mu_1\lambda_1+\mu_2\lambda_2)}{\left(Q+L\mu_1\lambda_1 E_r{}^{\frac{w_1}{\mu_1\lambda_1}}+L\mu_2\lambda_2 E_r{}^{\frac{w_2}{\mu_2\lambda_2}}\right)^2}$$

Note

$$\frac{w_1}{\mu_1\lambda_1} = \frac{\frac{\lambda_1\mu_1}{\lambda_1\mu_1+\lambda_2\mu_2}}{\mu_1\lambda_1} = \frac{1}{\lambda_1\mu_1+\lambda_2\mu_2} = \frac{1}{D}$$

$$\frac{w_2}{\mu_2\lambda_2} = \frac{\frac{\mu_2\lambda_2}{\lambda_1\mu_1+\lambda_2\mu_2}}{\mu_2\lambda_2} = \frac{1}{\lambda_1\mu_1+\lambda_2\mu_2} = \frac{1}{D}$$

Then

$$\frac{\partial E\big(C(Q,r)\big)}{\partial Q} = -\frac{A\mathrm{D}}{\left(Q+L\mu_1\lambda_1 E_r{}^{\frac{1}{\mathrm{D}}}+L\mu_2\lambda_2 E_r{}^{\frac{1}{\mathrm{D}}}\right)^2}+\frac{1}{2}H-\frac{\left(G_1L\lambda_1 E_r{}^{\frac{1}{\mathrm{D}}}+G_2L\lambda_2 E_r{}^{\frac{1}{\mathrm{D}}}\right)\mathrm{D}}{\left(Q+L\mu_1\lambda_1 E_r{}^{\frac{1}{\mathrm{D}}}+L\mu_2\lambda_2 E_r{}^{\frac{1}{\mathrm{D}}}\right)^2}$$

$$= -\frac{A\mathrm{D}}{\left(Q+L(\mu_1\lambda_1+\ \mu_2\lambda_2)E_r{}^{\frac{1}{\mathrm{D}}}\right)^2}+\frac{1}{2}H-\frac{(G_1\lambda_1+G_2\lambda_2)LE_r{}^{\frac{1}{\mathrm{D}}}\mathrm{D}}{\left(Q+L(\mu_1\lambda_1+\ \mu_2\lambda_2)E_r{}^{\frac{1}{\mathrm{D}}}\right)^2}$$

$$= \frac{1}{2}H-\frac{A\mathrm{D}}{\left(Q+L\mathrm{D}E_r{}^{\frac{1}{\mathrm{D}}}\right)^2}-\frac{(G_1\lambda_1+G_2\lambda_2)LE_r{}^{\frac{1}{\mathrm{D}}}\,\mathrm{D}}{\left(Q+L\mathrm{D}E_r{}^{\frac{1}{\mathrm{D}}}\right)^2}$$

$$= \frac{1}{2}H-\frac{A\mathrm{D}+(G_1\lambda_1+G_2\lambda_2)LE_r{}^{\frac{1}{\mathrm{D}}}\,\mathrm{D}}{\left(Q+L\mathrm{D}E_r{}^{\frac{1}{\mathrm{D}}}\right)^2}$$

Let $Z_r = (E_r)^{\frac{1}{\mathrm{D}}} = e^{-\frac{r}{LD}}$

$$\frac{\partial E\big(C(Q,r)\big)}{\partial Q} = \frac{1}{2}H-\frac{A\mathrm{D}+(G_1\lambda_1+G_2\lambda_2)LZ_r\,\mathrm{D}}{(Q+L\mathrm{D}Z_r)^2}$$

Similarly, the partial derivative of the objective function with respect to r simplifies to

$$\frac{\partial E\left(C(Q,r)\right)}{\partial r} = H\left(1 - w_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} - w_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right) - \frac{\left(\frac{G_1 w_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}}}{\mu_1} + \frac{G_2 w_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}}{\mu_2}\right)(\mu_1\lambda_1 + \mu_2\lambda_2)}{Q + L\mu_1\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + L\mu_2\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}}$$

$$+ \frac{\left(A + G_1 L\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + G_2 L\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)(\mu_1\lambda_1 + \mu_2\lambda_2)}{\left(Q + L\mu_1\lambda_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + L\mu_2\lambda_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)^2}\left(w_1 e^{-\frac{w_1 r}{L\mu_1\lambda_1}} + w_2 e^{-\frac{w_2 r}{L\mu_2\lambda_2}}\right)$$

$$= H\left(1 - w_1 E_r^{\frac{1}{\mathrm{D}}} - w_2 E_r^{\frac{1}{\mathrm{D}}}\right) - \frac{\left(\frac{G_1 w_1 E_r^{\frac{1}{\mathrm{D}}}}{\mu_1} + \frac{G_2 w_2 E_r^{\frac{1}{\mathrm{D}}}}{\mu_2}\right)(\mu_1\lambda_1 + \mu_2\lambda_2)}{Q + L\mu_1\lambda_1 E_r^{\frac{1}{\mathrm{D}}} + L\mu_2\lambda_2 E_r^{\frac{1}{\mathrm{D}}}} + \frac{\left(A + G_1 L\lambda_1 E_r^{\frac{1}{\mathrm{D}}} + G_2 L\lambda_2 E_r^{\frac{1}{\mathrm{D}}}\right)(\mu_1\lambda_1 + \mu_2\lambda_2)\left(w_1 E_r^{\frac{1}{\mathrm{D}}} + w_2 E_r^{\frac{1}{\mathrm{D}}}\right)}{\left(Q + L\mu_1\lambda_1 E_r^{\frac{1}{\mathrm{D}}} + L\mu_2\lambda_2 E_r^{\frac{1}{\mathrm{D}}}\right)^2}$$

Because $w_1 + w_2 = 1$, $\mu_1 * \lambda_1 + \mu_2 * \lambda_2 = D$, and $\mathrm{Z_r} = (E_r)^{\frac{1}{\mathrm{D}}}$

$$\frac{\partial E\left(C(Q,r)\right)}{\partial r} = H(1 - \mathrm{Z_r}) - \frac{(G_1\lambda_1 + G_2\lambda_2)\,\mathrm{Z_r}}{Q + LD\mathrm{Z_r}} + \frac{(A + G_1 L\lambda_1 \mathrm{Z_r} + G_2 L\lambda_2 \mathrm{Z_r})\,D\mathrm{Z_r}}{(Q + LD\mathrm{Z_r})^2}$$

APPENDIX 3: PROOF OF LEMMA 3

$$\frac{\partial E\big(C(Q,r)\big)}{\partial Q} = \frac{1}{2}H - \frac{A\mathrm{D} + (G_1\lambda_1 + G_2\lambda_2)LE_r^{\frac{1}{\mathrm{D}}}\,\mathrm{D}}{\left(Q + L\mathrm{D}E_r^{\frac{1}{\mathrm{D}}}\right)^2} = 0$$

In order words,

$$A * \mathrm{D} + (G_1\lambda_1 + G_2\lambda_2)LE_r^{\frac{1}{\mathrm{D}}}\,\mathrm{D} = \frac{1}{2}H\left(Q + L\mathrm{D}E_r^{\frac{1}{\mathrm{D}}}\right)^2$$

Since $Z_r = (E_r)^{\frac{1}{\mathrm{D}}}$

$$A\mathrm{D} + (G_1\lambda_1 + G_2\lambda_2)L\mathrm{Z_r}\mathrm{D} = \frac{1}{2}HQ^2 + H\mathrm{Q}L\mathrm{D}\mathrm{Z_r} + \frac{1}{2}H(L\mathrm{D}\mathrm{Z_r})^2$$

Thus,

$$H(L\mathrm{D})^2\mathrm{Z_r^2} + 2(H\mathrm{Q} - (G_1\lambda_1 + G_2\lambda_2)\,)L\mathrm{D}\mathrm{Z_r} + HQ^2 - 2AD = 0$$

Similarly, let

$$\frac{\partial E\big(C(Q,r)\big)}{\partial r} = H(1 - \mathrm{Z_r}) - \frac{(G_1\lambda_1 + G_2\lambda_2)\,\mathrm{Z_r}}{Q + LD\mathrm{Z_r}} + \frac{(A + G_1L\lambda_1\mathrm{Z_r} + G_2L\lambda_2\mathrm{Z_r})\,D\mathrm{Z_r}}{(Q + LD\mathrm{Z_r})^2} = 0$$

Thus,

$$H(1 - \mathrm{Z_r})(Q + LD\mathrm{Z_r})^2 - \left(\frac{G_1w_1\mathrm{Z_r}}{\mu_1} + \frac{G_2w_2\mathrm{Z_r}}{\mu_2}\right) D(Q + LD\mathrm{Z_r}\,) + (A + G_1L\lambda_1\mathrm{Z_r} + G_2L\lambda_2\mathrm{Z_r})D\mathrm{Z_r} = 0$$